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Annex No. 11 to the MU Directive on Habilitation Procedures and Professor Appointment Procedures

HABILITATION THESIS REVIEWER'S REPORT

Masaryk University	
Faculty	Faculty of Science
Procedure field	Mathematics – Applied Mathematics
Applicant	RNDr. Dana Černá, Ph.D.
Applicant's home unit, institution	Technical University in Liberec
Habilitation thesis	Wavelets on the Interval and Their Applications
Reviewer	Prof. Bin Han
Reviewer's home unit, institution	University of Alberta, Edmonton, Canada

One of the key challenging issues for wavelet methods in computational mathematics is to adapt Riesz/biorthogonal wavelets to the unit interval [0,1] with desired properties such as vanishing moments and small condition numbers. This excellent thesis made significant contributions to this important topic by constructing several spline Riesz wavelets on [0,1] with short support, small condition numbers and high vanishing moments for sparsity. In particular, employing very impressive analysis techniques, the candidate is able to adapt several spline biorthogonal wavelets whose dual wavelets have infinite support to the unit interval with extremely short support and very small condition numbers. Such constructions are innovative and are very appealing for their applications in numerical solutions of partial differential equations. The effectiveness of the constructed wavelets has been demonstrated by their applications to wavelet-Galerkin method and adaptive wavelet method. Overall, the thesis is excellent with high quality and the results greatly advanced the applications of wavelet methods to computational mathematics by constructing many impressive desired wavelets and multiwavelets on the unit interval [0,1], which also have a wide scope of applications in other areas such as applied mathematics and engineering.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer).

1) I think a few pages as well as references at the end of the second paper are missing. Also, page 16 of Chapter 1: Substituting (1.64) into (1.55), which should be (1.65).
2) Since many wavelets can be used in adaptive wavelet method which outperforms wavelet-Galerkin method, for comparison purposes, what are the most important properties of your constructed wavelets (e.g. short support, high vanishing moments, small condition numbers etc.) making adaptive wavelet method more effective?
3) Because adaptive wavelet method assumes that we can compute wavelet coefficients to any scale levels with accuracy, are there any hidden (or overhead) costs in adaptive wavelet method, comparing with wavelet-Galerkin method?

Conclusion

The habilitation thesis entitled Wavelets on the Interval and Their Applications by Dana Černá **fulfils** requirements expected of a habilitation thesis in the field of Mathematics – Applied Mathematics

Date: Jan. 15, 2020 Signature:

Habilitation Thesis: Wavelets on the Interval and Their Applications

By Dana Černá

As the major sparse multiscale representation system, wavelets have many successful applications such as image processing and numerical solutions to partial differential equations. To apply wavelet methods for numerical solutions of partial differential equations with various boundary conditions, the key issue is to adapt wavelets on the real line to bounded domains such as the unit interval so that the adapted wavelets preserve several desirable properties: the constructed wavelets on $[0, 1]$ have

(1) high vanishing moments for sparse coefficient matrices.

(2) short support and few boundary wavelets for simple implementation.

(3) the ability to satisfy various boundary conditions such as homogeneous Dirichlet boundary condition.

(4) small condition numbers of the coefficient matrices.

This habilitation thesis concentrates on constructing various desired wavelet bases on the unit interval and their applications to various partial differential equations and integral equations. Overall, this habilitation thesis is of high quality and makes a significant contribution for wavelet applications in computational mathematics. In particular, the constructed wavelets enjoy many highly desired properties such as short support, high vanishing moments and small condition numbers. These properties make such wavelets highly applicable in numerical solutions of various partial differential equations.

Wavelet methods (e.g., wavelet Galerkin/collocation method and adaptive wavelet method) provide several key advantages over traditional numerical methods such as finite element and finite difference methods. Short support and smoothness of wavelets offer good spatial and frequency localization. After a simple renormalization of elements, wavelets provide unconditional bases in many Sobolev spaces leading to uniformly bounded condition numbers. Vanishing moments of wavelets guarantee sparse multiscale representations so that the resulting coefficients matrices are sparse or quasi-sparse. The abstract theory of wavelet applications such as wavelet Galerkin method and adaptive wavelet method has been well developed in the literature now. For boundary value problems in a bounded domain, the classical wavelets on the real line have to be adapted to the unit interval $[0, 1]$ while preserve all their desired properties such as Riesz wavelet property, polynomial reproduction property, short support, and vanishing moments. Adapting wavelets on the real line to the unit interval while preserving all the desired properties is the key for the wavelet methods to many practical problems in computational mathematics. Despite a lot of effort in the literature on this topic for the past decades, this problem, in particular a general method on this problem, has not been satisfactorily resolved so far. This habilitation thesis makes a significant contribution in this direction for constructing several wavelets on $[0,1]$ which are highly applicable in computational mathematics.

Basically, the main body of this thesis consists of the eight papers addressing the problem of constructing desirable wavelets on $[0,1]$ (and their tensor product wavelets in rectangular domains) for numerical solutions of partial differential equations. The construction of wavelets on $[0,1]$ in this thesis can be classified into two groups.

(1) Construction of wavelets on $[0,1]$ from compactly supported biorthogonal wavelets on the real line. That is, both the primal and dual wavelets have compact support. These results are published in several papers. In particular, the wavelets on the unit interval with or without complementary boundary conditions derived from quadratic or cubic spline wavelets have much smaller condition numbers than other known constructions in the literature. This leads to improved performance in their adaptive wavelet method in numerical algorithms. The constructed wavelets on $[0,1]$ appear to be very interesting in computational mathematics and image processing.

(2) Construction of wavelets on $[0,1]$ (as well as on $[0,1]^d$ using tensor product) from biorthogonal wavelets on the real line such that only the primal wavelet has compact support while the dual wavelet has infinite support. Comparing with biorthogonal wavelets having compactly supported dual wavelets, allowing the dual wavelets to have infinite support, the primal wavelets can be made to have extremely short support and many desired properties. With extremely short support, the constructed wavelets on $[0,1]$ are very attractive for their applications to numerical solutions of various partial differential equations. Because the dual wavelets have infinite support, adapting the primal wavelet into the unit interval becomes a much harder problem than the approach in item (1). The main results on this topic are published in several papers for quadratic splines, cubic splines and Hermite cubic splines. The constructed wavelets have the smallest support with respect to the order of vanishing moments. The constructed wavelets have smaller support and condition numbers in comparison with the constructed wavelets in approach (1).

On one hand, the several papers in item (1) are interesting on constructing wavelets on $[0,1]$ from compactly supported biorthogonal wavelets. On the other hand, the constructed wavelets on $[0,1]$ in item (2) are very exciting, because such wavelets enjoy almost all the desirable properties for their applications to computational mathematics. In particular, the reviewer is deeply impressed by the candidate ability in proving the Riesz basis properties of such wavelets in item (2): The proofs involve a lot of complicated and subtle calculation and estimate on the operator norms of iterated matrices in (1.19) of Theorem 4. This shows the author's excellent background in mathematical analysis and ability in establishing the desired estimates/inequalities to reach the conclusion on the Riesz basis property. The candidate also provides many examples of the applications of the constructed wavelets on $[0, 1]$ to numerical mathematics and numerical partial differential equations to demonstrate their effectiveness.

Overall, this is an excellent thesis with high quality. The results in the thesis greatly advance the applications of wavelets methods to computational mathematics by constructing many desired wavelets and multiwavelets on the unit interval. Such constructed wavelets have a wide scope of applications in applied mathematics and engineering.

A few minor issues of the thesis for possible further improvements are as follows.

1. Page 91. The second paper: Cubic spline wavelets with complementary boundary conditions. I think a few pages as well as references at the end of the paper are missing.

2. Page 16 of Chapter 1, right below (1.65): Substituting (1.64) into (1.55). Here (1.55) should be (1.65).

3. Chapter 1 is very helpful for readers and is well written. Basically, there are two types of wavelet methods: Wavelet-Galerkin method in Section 1.6 and adaptive wavelet method in Section 1.7. Though adaptive wavelet method theoretically is optimal, it is a good idea to provide some information about their actual computational costs and comparison with wavelet-Galerkin method for concrete examples. In other words, in terms of actual computational costs, whether adaptive wavelet method is always better than wavelet-Galerkin method?

4. For the adaptive wavelet method, one often assumes that we can compute the wavelet coefficients to high scale levels with a desired accuracy. The scale level used in wavelet-Galerkin method is clear. So, it is a good idea to indicate what is the highest scale level that is used in adaptive wavelet method. In other words, a little bit more details on adaptive wavelet method will allow other readers to reproduce all the numerical examples reported in the included papers. Which particular properties of the constructed wavelets are the most important ones (short support, high vanishing moments, small condition numbers etc.) for the adaptive wavelet method?