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HABILITATION THESIS REVIEWER'S REPORT

Masaryk University	
Faculty	Faculty of Science
Procedure field	Mathematics - Geometry
Applicant	Yaroslav Bazaykin, Ph.D., D.Sci.
Applicant's home unit, institution	University of Hradec Kralove, Faculty of Science, Department of Mathematics
Habilitation thesis	Noncompact Riemannian manifolds with special holonomy
Reviewer	Prof. Andrew Swann, BA, MA, D.Phil.
Reviewer's home unit, institution	Department of Mathematics, Aarhus University, Denmark

The thesis consists of 6 publish papers (A – F) of the author, from 2004 to 2011. Four of these are single author papers, the other two (C and D) are joint with E. G. Malkovich. In addition the thesis contains an introduction introducing the general area of research and a brief summary of the papers.

The papers lie in the area of differential geometry, more specifically study of Ricci-flat metrics. Given the Berger holonomy classification, which in principal is from 1955, but has long following history and to a certain extent is still not yet complete, of particular significance are metrics in dimension 8 with holonomy in Spin(7), and in dimension 7 with holonomy G2. In four dimensions, the holonomy group SU(2) = Sp(1) is of central importance. All these geometries have interest in, and interactions with, theoretical physics through gravitation and supersymmetry.

A central problem is to construct examples of such metrics, thus solutions of the vacuum Einstein equations. For holonomies G2 and Spin(7), first examples were found in the 1980's by Bryant, and Bryant & Salamon. In the mid 1990's Joyce constructed the first compact examples. Compact examples appear to necessarily be of a transcendental nature, so if one wishes to have explicit examples, then it is best to work in the non-compact realm. Mathematically, it is then natural to ask for examples where the metrics are complete; indeed the Bryant-Salamon examples have this character and their examples also have rich symmetry. In the late 1990's and early 2000's a number of mathematicians and physicists studied and constructed various solutions with symmetry of cohomogeneity one, meaning that principal orbits are of codimension one. This situation reduces the complicated partial differential equations from the holonomy condition, to systems of ordinary differential equations.

Papers A – D make significant contributions to this latter approach, and show that the ideas apply in rather wider contexts. In particular, detailed correct mathematical arguments are given to show existence of solutions that were only conjectured numerically in the physics literature. The techniques involve understanding the ODE system, first noting homogeneity properties, and then using this to reduce to an ODE system on a compact space. This new system is still not simple, straightforward application of standard study of fixed points etc. can not be applied directly. Instead discrete symmetries of the system, combined with blow-up techniques, are used to arrive at existence results for complete metrics of certain type, and to construct families of solutions (with one or two parameters). This provides significant extensions to the previously known results, and includes interesting phenomena, whereby certain classes of solutions are shown to necessarily have certain asymptotic behaviours (asymptotically locally Euclidean, asymptotically conincal), that other authors would put in as assumptions.

Three of these papers apply to the Spin(7) case, the fourth is for G2 metrics. The results generalise the applicability of the cohomogeneity one approach, by instead studying structures on (essentially) three-dimensional bundles over four-dimensional orbifold bases. The important idea is to mediate via so-called 3-Sasakian structures; locally the Spin(7) set-up is the product of a 3-Sasakian manifold with an interval, the 3-Sasaki data varying in a specified way with respect the parameter on the interval. Due to the work of Boyer & Galicki on 3-Sasakian manifolds, which resulted in a large survey paper 1999, and a monograph 2008, much is known about such manifolds, and Bazaykin's work then gives many examples in the orbifold category, which is also of interest to the physics.

Papers E and F are more concerned with four-dimensional geometry. To a certain extent paper F, makes paper E obsolete. Paper F studies directly Ricci-flat metrics in dimension four on complex line bundles over a two-dimensional sphere. The potential model is T^*CP(1) which carries the Eguchi-Hanson metric; a model from relativity is the Kerr metric. An explicit family of solutions is written down depending on a rational parameter, in general with two orbifold points. In paper F, it is realised that these metrics have holonomy SU(2), together with the symmetry consideration this means that they are in fact given by the Gibbons-Hawking construction (1970's) by varying certain parameters. Paper F goes on to use these metrics to generalise the Kummer construction of K3 surfaces, in a way comparable with an approach originally suggested by Page (1978). As in the Kummer construction, the idea is to consider a quotient of four-torus T4 by a discrete group, and to glue in Ricci-flat metrics at the singularities, and then to deform to smooth Ricci-flat solution. This method was for many years technically hard to implement correctly, there is a much better understanding of this now (including papers from the last two years). Bazaykin cleverly moves the analysis to use result of Joyce, by taking a product with a T3 and using Joyce's technical theorems to get existence of metrics of holonomy G2; observations on the fundamental group then yield the four-dimensional solutions. This paper F deserves to be better known than it is.

The introduction of the thesis, provides some historical background to and a summary of the six papers. This is done quite well, but it also has some weakness. There are a number of linguistic errors in the text and the abstract, including simple spelling mistakes, that should have been corrected before submission. What I miss most is discussion of the newer context of the material – the most recent reference is a survey paper from 2012. This is an area that is under significant development, and an indication from the author of how his work fits into that would have been good.

In conclusion, the thesis demonstrates significant contributions to the theory of Ricci-flat metrics of special holonomy and the construction of solutions. This fits in to and contributes to a significant area of differential geometry that has import interactions with physics.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)

What is the extent of the author's contribution to the two joint papers in the thesis?

What are the reasons for the author selecting the chosen articles from his list of publications, rather than other strong papers he has there?

What developments have there been in the study of metrics of special holonomy since 2011, particularly in relation to directions covered by the thesis?

To what other problems in Riemannian geometry have these ODE techniques been applied to?

On page 7, you state hyperkähler metrics are easier to construct than Calabi-Yau metrics and then cite Joyce's large book. What do you mean exactly, and which parts of Joyce's book are the relevant ones?

Conclusion

The habilitation thesis entitled "Noncompact Riemannian manifolds with special holonomy" by Yaroslav Bazaykin **fulfils** requirements expected of a habilitation thesis in the field of Mathematics - Geometry.

Date: 6th Oct. 2020 Signature: