Commentary on Habilitation Thesis

Wavelets on the Interval and Their Applications

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Wavelet bases and the fast wavelet transform are a powerful and useful tool for signal and image analysis, detection of singularities, data compression, and also for the numerical solution of partial differential equations, integral equations, and integro-differential equations. One of the most important properties of wavelets is that they have vanishing moments. Vanishing wavelet moments ensure the so-called compression property of wavelets. This means that integrals of a product of a function and a wavelet decay exponentially, dependent on the level of the wavelet if the function is smooth enough in the support of the wavelet. This enables the obtainment of sparse representations of functions as well as sparse representations of some operators, see e.g. [1, 14, 20].

There are two main classes of wavelet based methods for the numerical solution of operator equations. The first method is the wavelet-Galerkin method. Due to vanishing moments, the wavelet-Galerkin method leads to sparse matrices not only for differential equations but also for integral and integro-differential equations while the Galerkin method with the standard B-spline basis leads to full matrices if the equation contains an integral term. Another important property of wavelet bases is that they form Riesz bases in certain spaces, such as Lebesgue, Sobolev or Besov spaces. Due to this property, the diagonally preconditioned matrices arising from discretization using the Galerkin method with wavelet bases have uniformly bounded condition numbers for many types of operator equations.

The second class of methods are adaptive wavelet methods. We focus on adaptive wavelet methods that were originally designed in [15, 16] and later modified in many papers [17, 18, 19]. For a large class of operator equations, both linear and nonlinear, it was shown that these methods converge and are asymptotically optimal in the sense that the storage and the number of floating point operations, needed to resolve the problem with desired accuracy, depend linearly on the number of parameters representing the solution. Moreover, the method enables higher-order approximation if higher-order spline-wavelet bases are used. The solution and the right-hand side of the equation have sparse representations in a wavelet basis, i.e. they can be represented by a small number of numerically significant parameters. Similarly as in the case of the wavelet-Galerkin method, the differential and integral operators can be represented by sparse or quasi-sparse matrices. For a large class of problems, the matrices arising from a discretization using wavelet bases can be simply preconditioned by a diagonal preconditioner, and the condition numbers of these preconditioned matrices are uniformly bounded. For more details about adaptive wavelet methods, see [3, 15, 16, 17, 18, 19, 20].

The first wavelet methods used orthogonal wavelets, e.g. Daubechies wavelets or coiflets. Their disadvantages are that the most orthogonal wavelets are usually not known in an explicit form and their smoothness is typically dependent on the length of the support. In contrast, spline wavelets are known in a closed form, are smoother, and have shorter support than orthogonal wavelets with the same polynomial exactness and the same number of vanishing moments. Therefore, they are preferable in numerical methods for operator equations.

The objectives of the habilitation thesis are constructions of new spline-wavelet bases on the interval and product domains adapted to various types of boundary conditions, which outperform the existing bases of the same type with respect to the efficiency of the numerical methods used for solution of differential and integro-differential equations. More precisely, to construct wavelet bases such that the wavelet-Galerkin method and adaptive wavelet methods using these bases lead to better conditioned discretization matrices, sparser matrices, smaller numbers of iterations, smaller numbers of parameters representing the solution with a desired accuracy, etc., than for other bases of the same type. The aim is also to illustrate the efficiency and applicability of the constructed bases on applications.

The thesis is conceived as a collection of the eight previously published articles [4, 5, 6, 7, 8, 10, 12] supplemented by commentary.

The papers [4, 5, 8] are focused on constructions of well-conditioned biorthogonal spline wavelet bases on the interval where both primal and dual wavelets have compact support. In [6, 7, 12, 13] we do not require local support of dual wavelets, which enables us to construct wavelet bases that have smaller support and have significantly smaller condition number than wavelet bases with local duals. Moreover, their construction is significantly simpler than constructions of wavelets with local duals, which are typically quite long and technical. In [11], we constructed wavelets that are orthogonal to piecewise polynomials of degree at most seven on a uniform grid. Due to this property, matrices arising from discretization of second-order differential equations with coefficients that are piecewise polynomials of degree at most four on uniform grids are sparse. We use the constructed bases for solving various types of operator equations, e.g. Poisson's equation, the Helmholtz equation, fourth-order differential equations, and the Black-Scholes equation with two state variables. We also applied the constructed bases for option pricing under Kou's double exponential jump-diffusion option pricing model. Other applications are presented in Chapter 2.

Author contributions

I am the sole author of the article [13]. I prepared the whole manuscript and performed all numerical experiments presented in this paper. The other articles [4, 5, 6, 7, 8, 10, 12] are the results of collaborative work with Václav Finěk. I am the main and corresponding author of all these papers, and my overall contribution is about 60% for the papers [4, 5, 6, 7, 8, 10] and about 90% for the paper [12]. I came up with the main ideas and proposed a basic concept. Then both authors participated together in all stages of research connected with the papers. They established theory, prepared manuscript, created programs, performed numerical experiments, and revised manuscript.

References

[1] Beylkin, G.; Coifman, R.; Rokhlin, V.: *Fast wavelet transforms and numerical algorithms I*, Commun. Pure Appl. Math. **44**, 1991, pp. 141–183.

[2] Calderón, L.; Martín, M.T.; Vampa, V.: *Bases de wavelets B-splines en el intervalo con condiciones de dirichlet homogéneas*, in: V Jornadas de Investigación, Transferencia y Extensión de la Facultad de Ingeniería, La Plata, 2019, pp. 235–240.

[3] Černá, D.: *Biorthogonal Wavelets*, Ph.D. thesis. Charles University, Prague, 2008.

[4] Černá, D.; Finěk, V.: *Construction of optimally conditioned cubic spline wavelets on the interval*, Adv. Comput. Math. **34(2)**, 2011, pp. 219–252.

[5] Černá, D.; Finěk, V.: *Cubic spline wavelets with complementary boundary conditions*, Appl. Math. Comput. **219**, 2012, pp. 1853–1865.

[6] Černá, D.; Finěk, V.: *Quadratic spline wavelets with short support for fourth-order problems*, Result. Math. **66**, 2014, pp. 525–540.

[7] Černá, D.; Finěk, V.: *Cubic spline wavelets with short support for fourth-order problems*, Appl. Math. Comput. **243**, 2014, pp. 44–56.

[8] Černá, D.; Finěk, V.: *Wavelet basis of cubic splines on the hypercube satisfying homogeneous boundary conditions*, Int. J. Wavelets Multi. **13(3)**, 2015, article No. 1550014.

[9] Černá, D.; Finěk, V.: *On a sparse representation of an n-dimensional Laplacian in wavelet coordinates,* Result. Math. **69**, 2016, pp. 225–243.

[10] Černá, D.; Finěk, V.: *Sparse wavelet representation of differential operators with piecewise polynomial coefficients*, Axioms **6**, 2017, article No. 4.

[11] Černá, D.: *Adaptive wavelet method for pricing two-asset Asian options with floating strike,* in: 43rd International Conference Applications of Mathematics in Engineering and Economics, Bulgaria, 2017; AIP Conference Proceedings **1910**, American Institute of Physics, New York, 2017, article No. 030001.

[12] Černá, D.; Finěk, V.: *Quadratic spline wavelets with short support satisfying homogeneous boundary conditions*, Electron. Trans. Numer. Anal. **48**, 2018, pp. 15–39.

[13] Černá, D.: *Cubic spline wavelets with four vanishing moments on the interval and their applications to option pricing under Kou model*, Int. J. Wavelets Multiresolut. Inf. Process. **17**, 2019, article No. 1850061.

[14] Cohen, A.: *Numerical Analysis of Wavelet methods*, Studies in Mathematics and its Applications **32**, Elsevier, Amsterdam, 2003.

[15] Cohen, A.; Dahmen, W.; DeVore, R.: *Adaptive wavelet schemes for elliptic operator equations - convergence rates*, Math. Comput. **70**, 2001, pp. 27–75.

[16] Cohen, A.; Dahmen, W.; DeVore, R.: *Adaptive wavelet methods II - beyond the elliptic case*, Found. Math. **2**, 2002, pp. 203–245.

[17] Dijkema, T.J.: *Adaptive Tensor Product Wavelet Methods for Solving PDEs*, Ph.D. thesis, University Utrecht, 2009.

[18] Hilber, N.; Reichmann, O.; Schwab, C.; Winter, C.: *Computational Methods for Quantitative Finance*, Springer, Berlin, 2013.

[19] Stevenson, R.: *Adaptive solution of operator equations using wavelet frames*, SIAM J. Numer. Anal. **41**, 2003, pp. 1074–1100.

[20] Urban, K.: *Wavelet Methods for Elliptic Partial Differential Equations,* Oxford University Press, Oxford, 2009.