MASARYK UNIVERSITY
FACULTY OF SCIENCE


# Construction of Wavelets

Habilitation Thesis


**Václav Finěk**



**2015**

# Contents

# List of Symbols

| | |
|---|---|
| $\mathbb{N}$ | the set of all positive integers |
| $\mathbb{N}_0$ | the set of all nonnegative integers |
| $\mathbb{Z}$ | the set of all integers |
| $\mathbb{R}$ | the set of all real numbers |
| $\overline{\Omega}$ | the closure of the set $\Omega$ |
| $\lvert\cdot\rvert$ | the absolute value |
| $\delta_{i,j}$ | the Kronecker delta, $\delta_{i,i} := 1$, $\delta_{i,j} := 0$ for $i \neq j$ |
| span | the linear span |
| supp | the support |
| $\kappa(\cdot)$ | the spectral condition number of a matrix |
| $C \lesssim D$ | means that $C$ can be bounded by a multiple of $D$ independently of parameters on which they may depend |
| $L^p(0,1)$ | the space of square integrable functions |
| $H^s(0,1)$ | the Sobolev space of order $s \in \mathbb{R}$ on $(0,1)$ |
| $H_0^1(0,1)$ | the Sobolev space of $H^1$ functions satisfying homogeneous Dirichlet boundary conditions |
| $C^m(\mathbb{R})$ | $m \in \mathbb{N}_0$, the space of $m$-times continuously differentiable functions |
| $l^2(\mathcal{J})$ | $l^2(\mathcal{J}) := \left\{\mathbf{v} : \mathcal{J} \to \mathbb{R}, \sum_{\lambda\in\mathcal{J}} \lvert\mathbf{v}_\lambda\rvert^2 < \infty\right\}$ |
| $\Pi_m(0,1)$ | the space of all algebraic polynomials on $(0,1)$ of degree less or equal to $m \in \mathbb{N}_0$ |
| $\lVert\cdot\rVert$ | $L^2$-norm |
| $\lVert\cdot\rVert_H$ | a norm on some space $H$ |
| $\lvert\cdot\rvert_{H^s(0,1)}$ | the seminorm on $H^s(0,1)$ |
| $\langle\cdot,\cdot\rangle$ | $L^2$-inner product or a dual form |
| $\langle\cdot,\cdot\rangle_H$ | an inner product in $H$ |

| | |
|---|---|
| $j_0$ | the coarsest level in a multiresolution analysis in a given context |
| $\lambda$ | a wavelet index $\lambda := (j, k)$ |
| $\phi$ ($\tilde{\phi}$) | a primal (dual) scaling function |
| $\psi$ ($\tilde{\psi}$) | a primal (dual) wavelet |
| $\hat{f}$ | the Fourier transform of the function $f$ |
| $f_{j,k}(x)$ | the translation and the dilatation of the function $f_{j,k}(x) := f(2^j x - k)$ |

# Introduction

We present here some obtained results concerning constructions of well-conditioned wavelet bases. This work consists of three chapters. In the first chapter, we shortly introduce wavelet bases on the real line generated by one wavelet and by one scaling function. We start with properties of Riesz bases. Then we continue with a multiresolution analysis and with orthonormal and biorthogonal wavelets. Consequently we introduce oblique projections, the discrete wavelet transform, approximation properties of wavelets and we collect some important properties of B-splines which are often used as primal scaling functions.
In the second chapter, we introduce wavelet bases on the bounded interval. They are usually constructed from wavelets on the real line. The main idea is to retain most of the inner functions, i.e. the scaling functions and wavelets whose supports is contained in the interval, and to construct appropriate boundary scaling functions and wavelets separately. At the same time the important properties of wavelets should be preserved such as a Riesz basis property, a smoothness, a local support of basis functions and a polynomial exactness of the wavelet basis. Unlike the first chapter, we consider in the second chapter wavelet systems generated by many wavelets and by many scaling functions. Wavelets can be even different at different decomposition levels. The second chapter contains basic definitions, a derivation of a multiscale transform, theorems which can be used to prove that constructed basis is a Riesz basis, and finally we show that condition numbers of stiffness matrices arising from discretization of elliptic partial differential equations by wavelets depend on Riesz constants of a wavelet basis.
In the third chapter we present selected results which were published in the following five papers:

- Černá, D.; Finěk, V.; Najzar, K.: *On the exact values of coefficients of Coiflets*, Cent. Eur. J. Math. **6(1)**, (2008), pp. 159-170. My contribution to this paper was 60%.

- Černá, D.; Finěk, V.: *Construction of optimally conditioned cubic spline wavelets on the interval*, Adv. Comput. Math. **34(2)**, (2011), pp. 219-252. My contribution to this paper was 40%.

- Černá, D.; Finěk, V.: *Cubic Spline Wavelets with Complementary Boundary Conditions*, Appl. Math. Comput. **219**, (2012), pp. 1853-1865. My contribution to this paper was 40%.

- Černá, D.; Finěk, V.: *Wavelet basis of cubic splines on the hypercube satisfying homogeneous boundary conditions*, Int. J. Wavelets Multi. **13(3)**, (2015), pp. 1550014/1-21. My contribution to this paper was 40%.

- Černá, D.; Finěk, V.: *On a sparse representation of a n-dimensional Laplacian in wavelet coordinates*, Result. Math., DOI 10.1007/s00025-015-0488-5, (2015). My contribution to this paper was 60%.

In the paper "On the Exact Values of Coefficients of Coiflets" [14], we proposed a system of necessary conditions which is redundant free and more simple than other known systems due to elimination of some quadratic (orthonormality) conditions, thus a computation of scaling coefficient of coiflets is substantially simplified and enables to find the exact values of the scaling coefficients up to filters of the length 8 and two further with filters of the length 12. For scaling coefficients of coiflets with filters of the length 14 we obtained two quadratic equations, which can be transformed to polynomial of degree 4 and there is an algebraic formula to solve them. For larger filters up to filters of the length 20, we were able to find all possible solutions by employing a Gröbner basis method.

In the paper "Construction of Optimally Conditioned Cubic Spline Wavelets on the Interval" [7], we constructed spline wavelet bases on the interval with condition numbers which are close to condition numbers of spline wavelet bases on the real line. Both primal and dual functions are compactly supported. Constructed cubic wavelet bases have improved condition numbers in comparison with previous constructions of the same type. Furthermore, we showed that the constructed wavelets form indeed a Riesz basis for the space $L^2(0,1)$ and for the Sobolev space $H^s(0,1)$ for a certain range of $s$. Finally, we adapted primal bases to homogeneous Dirichlet boundary conditions of the first order and we compared quantitative properties of the constructed bases and the efficiency of an adaptive wavelet scheme for several spline wavelet bases to demonstrate a superiority of our construction.

In the paper "Cubic Spline Wavelets with Complementary Boundary Conditions" [8], we constructed a new stable cubic spline wavelet basis on the interval with six vanishing moments. The proposed basis satisfies complementary boundary conditions of the second order i.e. the primal basis functions are adapted to homogeneous Dirichlet boundary conditions of the second order, while the dual wavelet basis preserves the full degree of polynomial exactness. Moreover, we proposed further decomposition of the scaling basis at the coarsest level. It leads to improved Riesz condition numbers of the proposed basis. Finally, we presented quantitative properties of the proposed basis and we compared them with some other cubic spline wavelet bases to show superiority of our construction. Numerical examples were presented for the two-dimensional biharmonic equation.

In the paper "Wavelet Basis of Cubic Splines on the Hypercube Satisfying Homogeneous Boundary Conditions" [12], we constructed new cubic spline wavelet basis on the hypercube that is well-conditioned, adapted to homogeneous Dirichlet boundary conditions and the wavelets have two vanishing moments. Unlike our construction proposed in [7], we do not require a compact support for dual functions which enables to construct primal functions with better properties. The advantage of our construction is that the support of wavelets is shorter, Riesz condition numbers are smaller and another advantage is also a simple construction. Then stiffness matrices arising from discretization of elliptic problems using proposed wavelets have uniformly bounded condition numbers and these condition numbers are small. It leads in combination with the shorter support of wavelets to more efficient numerical solvers. Finally, we presented quantitative properties of the constructed basis and we provided a numerical example to show an efficiency of Galerkin method using constructed basis.

In the paper "On a Sparse Representation of a $n$-dimensional Laplacian in Wavelet Coordinates" [13], we constructed a wavelet basis based on Hermite cubic splines with respect to which both the mass matrix and the stiffness matrix corresponding to one dimensional Poisson equation are sparse. While stiffness matrices in wavelet coordinates are usually only quasi sparse. Then, matrix-vector multiplication can be performed exactly with linear complexity for any second order PDEs with constant coefficients. Moreover, the proposed basis is very well-conditioned for low decomposition levels. Small condition numbers for low decomposition levels and a sparse structure of stiffness matrices are kept for any second order PDEs with constant coefficients, which are well-conditioned in the sense of (2.7), and moreover they are independent of the space dimension. Further, we proved that the constructed basis is a Riesz basis and computed condition numbers for model problems and compared them with condition numbers for a similar wavelet basis proposed in [29].

All paper presented in this thesis comes from a collaboration with my colleague Dana Černá. I would like to thank her for this fruitful collaboration, and look forward to its continuation.

# Chapter 1

# Wavelets on the Real Line

In this chapter, we shortly introduce wavelet bases on the real line generated by one wavelet and by one scaling function. First, we introduce an important definition of a Riesz basis which is a generalization of an orthonormal basis.

## 1.1 Riesz Bases

**Definition 1.** A family $\{e_k\}_{k\in\mathbb{Z}}$ is called a *Riesz basis* of a Hilbert space $H$, if and only if it spans $H$, i.e. all finite linear combinations of the $e_k$ are dense in $H$, and if there exist constants $c, C$ such that $0 < c \leq C$ and

$$c\left(\sum_{k\in\mathbb{Z}} |x_k|^2\right)^{1/2} \leq \left\|\sum_{k\in\mathbb{Z}} x_k e_k\right\|_H \leq C\left(\sum_{k\in\mathbb{Z}} |x_k|^2\right)^{1/2} \qquad \forall \{x_k\} \in l^2(\mathbb{Z}). \tag{1.1}$$

The constants $c, C$ are called *Riesz bounds.*

It is well known that any orthonormal basis satisfies (1.1) with $c = C = 1$. Riesz bases have many useful properties of orthonormal bases without requiring orthonormality. The condition (1.1) can be interpreted as ensuring stability of the reconstruction of an arbitrary element $x \in H$ from its coefficients $\{x_k\}$ in the sense that small roundoff errors in the computation of the coefficients $x_k$ can not lead to a large error in the reconstruction. The main properties of Riesz bases are summarized in the following theorem.

**Theorem 2.** *Let $\{e_k\}_{k\in\mathbb{Z}}$ be a Riesz basis in a separable Hilbert space $H$ and let the operator $T : l^2(\mathbb{Z}) \to H$ be defined by*

$$T : \{c_k\}_{k\in\mathbb{Z}} \mapsto \sum_{k\in\mathbb{Z}} c_k e_k.$$

*Then*

- *The series $\sum_{k\in\mathbb{Z}} c_k e_k$ converges unconditionally in $H$, i.e. its terms can be arbitrarily permuted without affecting the convergence, if and only if $\{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$.*

- *Any $x \in H$ can be decomposed in a unique way according to*

$$x = \sum_{k\in\mathbb{Z}} c_k e_k \quad \text{with} \quad \{c_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z}).$$

- *$T$ is an isomorphism from $l^2(\mathbb{Z})$ to $H$.*

- *There exists a unique biorthogonal Riesz basis $\{\tilde{e}_k\}_{k\in\mathbb{Z}}$ in $H$, i.e. $\{\tilde{e}_k\}_{k\in\mathbb{Z}}$ is a Riesz basis and $\langle e_k, \tilde{e}_l\rangle_H = \delta_{k,l}$. This basis is defined by*

$$\tilde{e}_k = (TT^*)^{-1} e_k,$$

*where $T^*$ denotes the adjoint mapping to $T$.*

- *There exists constants $0 < c \leq C$ such that*

$$c\,\|x\|_H^2 \leq \sum_{k\in\mathbb{Z}} \left|\langle x, e_k\rangle_H^2\right| \leq C\,\|x\|_H^2 \qquad \forall x \in H.$$

Further details can be found in [18]. The next theorem gives equivalent conditions for $\{e_k\}_{k\in\mathbb{Z}}$ to be a Riesz basis.

**Theorem 3.** *For a sequence $\{e_k\}_{k\in\mathbb{Z}}$ spanning a Hilbert space $H$, the following conditions are equivalent:*

- *$\{e_k\}_{k\in\mathbb{Z}}$ is a Riesz basis for $H$.*

- *The Gram matrix $\{\langle e_k, e_l\rangle\}_{k,l\in\mathbb{Z}}$ defines a bounded, invertible operator on $l^2(\mathbb{Z})$.*

- *$\{e_k\}_{k\in\mathbb{Z}}$ is a Bessel sequence, and there exists a biorthogonal sequence $\{f_k\}_{k\in\mathbb{Z}}$ which is also a Bessel sequence spanning $H$.*

The proof of this Theorem can be found in [15]. For completeness we provide also a definition of a Bessel sequence.

**Definition 4.** A sequence $\{e_k\}_{k\in\mathbb{Z}}$ in a separable Hilbert space $H$ is called a Bessel sequence if there exists a constant $C > 0$ such that

$$\left\|\sum_{k\in\mathbb{Z}} x_k e_k\right\|_H \leq C\left(\sum_{k\in\mathbb{Z}} |x_k|^2\right)^{1/2} \qquad \forall\{x_k\} \in l^2(\mathbb{Z}).$$

## 1.2 Wavelets

In this chapter, we consider $H = L^2(\mathbb{R})$. A function $\phi$ is called $L^2$-*stable* if $\{\phi(x-k)\}_{k\in\mathbb{Z}}$ is a Riesz basis of its span in $L^2(\mathbb{R})$. Now we can define a wavelet, a biorthogonal wavelet and an orthonormal wavelet.

**Definition 5.** A function $\psi \in L^2(\mathbb{R})$ is called a *wavelet* if the family of functions $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$, where $\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k)$, is a Riesz basis of $L^2(\mathbb{R})$. A dual wavelet $\tilde{\psi}$ is called *biorthogonal* to a (primal) wavelet $\psi$ if

$$\left\langle \psi_{j,k}, \tilde{\psi}_{i,l} \right\rangle = \delta_{i,j}\delta_{k,l} \qquad \forall i,j,k,l \in \mathbb{Z}.$$

The wavelet is called *orthonormal* if

$$\langle \psi_{j,k}, \psi_{i,l} \rangle = \delta_{i,j}\delta_{k,l} \qquad \forall i,j,k,l \in \mathbb{Z}.$$

**Example 6.** The simplest example of orthonormal wavelet is the Haar wavelet. The Haar wavelet is the function defined on the real line as

$$H(x) = \begin{cases} 1 & \forall x \in \left[0, \frac{1}{2}\right), \\ -1 & \forall x \in \left[\frac{1}{2}, 1\right], \\ 0 & \text{otherwise.} \end{cases}$$

It is well-known [42, 43] that the system $\left\{2^{j/2}H(2^j x - k)\right\}_{j,k\in\mathbb{Z}}$ is orthonormal in $L^2(\mathbb{R})$.

Wavelets are usually constructed with an assistance of a multiresolution analysis.

**Definition 7.** A sequence $\{V_j\}_{j\in\mathbb{Z}}$ of closed subspaces of $L^2(\mathbb{R})$ is called a *multiresolution analysis* if it satisfies the following conditions:

1) The sequence is nested, i.e.

$$V_j \subset V_{j+1} \qquad \forall j \in \mathbb{Z}.$$

2) The spaces are related to each other by dyadic scaling, i.e.

$$f(x) \in V_j \Leftrightarrow f(2x) \in V_{j+1} \qquad \forall j \in \mathbb{Z}.$$

3) The union of the spaces is dense, i.e.

$$\overline{\bigcup_{j\in\mathbb{Z}} V_j} = L^2(\mathbb{R}).$$

4) The intersection of the spaces is reduced to the set containing only the null function, i.e.

$$\bigcap_{j\in\mathbb{Z}} V_j = \{0\}.$$

5) There exists a *scaling function* $\phi \in V_0$ such that

$$\{\phi(x-k)\}_{k\in\mathbb{Z}}$$

is a Riesz basis of $V_0$.

If we construct an orthonormal basis than we require at the point 5) of the above definition that a scaling function $\phi$ forms an orthonormal basis of $V_0$. Spaces $V_j$ from the above definition are often called *principal shift-invariant spaces* and most of their properties can be studied by means of Fourier analysis.

**Example 8.** For the Haar system $\left\{2^{j/2}H(2^j x-k)\right\}_{j,k\in\mathbb{Z}}$, the scaling function is defined by [42, 43]:

$$\phi_H(x)=\begin{cases}1 & \forall x\in[0,1]\\ 0 & \text{otherwise.}\end{cases}$$

Consequences of the Riesz basis property:

- Since $V_0 \subset V_1$ and from Theorem 2, there exists a sequence $\{h_k\}_{k\in\mathbb{Z}} \in l^2(\mathbb{Z})$ such that
$$\phi(x)=\sum_{k\in\mathbb{Z}} h_k\phi(2x-k) \qquad \forall x\in\mathbb{R}. \tag{1.2}$$
This equation is called *refinement* or *scaling equation* and the coefficients $h_k$ are known as *scaling* or *refinement coefficients*. These coefficients will be used later in a discrete wavelet transform and also to a construction of dual wavelets.

- For each $j\in\mathbb{Z}$, the set $\left\{2^{j/2}\phi(2^j x-k)\right\}_{k\in\mathbb{Z}}$ is a Riesz basis of $V_j$ with Riesz bounds independent of $j$.

## 1.3 Biorthogonal wavelets

Now, let have two different scaling functions $\phi$ and $\tilde{\phi}$, which usually generate different multiresolution analyses $\{V_j\}_{j\in\mathbb{Z}}$, $\left\{\tilde{V}_j\right\}_{j\in\mathbb{Z}}$, and consequently also two different wavelet functions $\psi$, $\tilde{\psi}$. Then we have two sets of scaling coefficients while we have only one set of scaling coefficients in the case of orthonormal wavelets. Therefore biorthogonal wavelets provide more degrees of freedom in comparison with orthonormal wavelets and it is possible to construct primal wavelets with better properties. For instance, primal wavelets are usually smoother than orthonormal wavelets with the same length of the support.
Wavelet coefficients can be determined as

$$g_n=(-1)^n\,\tilde{h}_{1-n}, \qquad \tilde{g}_n=(-1)^n\,h_{1-n}, \tag{1.3}$$

where $h_n$ and $\tilde{h}_n$ are scaling coefficients corresponding to $\phi$ and $\tilde{\phi}$, respectively. Wavelets are then given by

$$\psi(x) = \sum_{n\in\mathbb{Z}} g_n \phi(2x-n), \qquad \tilde{\psi}(x) = \sum_{n\in\mathbb{Z}} \tilde{g}_n \tilde{\phi}(2x-n) \qquad \forall x \in \mathbb{R}. \tag{1.4}$$

The function $\phi$ is called a *primal* scaling function, the sequence $\{V_j\}_{j\in\mathbb{Z}}$ is called a *primal* multiresolution analysis and $\psi$ is a *primal* wavelet, while $\tilde{\phi}$, $\left\{\tilde{V}_j\right\}_{j\in\mathbb{Z}}$, and $\tilde{\psi}$ are called *dual*. Let us define

$$W_j = \overline{\operatorname{span}\{\psi_{j,k}, k\in\mathbb{Z}\}}, \qquad \tilde{W}_j = \overline{\operatorname{span}\left\{\tilde{\psi}_{j,k}, k\in\mathbb{Z}\right\}}.$$

The following lemma describes basic properties of biorthogonal wavelets.

**Theorem 9.** *Let sequences $\{V_j\}_{j\in\mathbb{Z}}$ and $\left\{\tilde{V}_j\right\}_{j\in\mathbb{Z}}$ be two multiresolution analyses with mutually biorthogonal scaling functions $\phi$ and $\tilde{\phi}$ so that $\left\langle \phi(x-k), \tilde{\phi}(x-l)\right\rangle = \delta_{k,l}$ for all $k,l\in\mathbb{Z}$. Further let wavelet coefficients be defined by (1.3) and primal and dual wavelets be defined by (1.4). Then*

- *$\psi(x) \in V_1$ and $\tilde{\psi}(x) \in \tilde{V}_1$.*
- *$\tilde{\psi}$ is biorthogonal to $\psi$, i.e.*

$$\left\langle \psi(x-k), \tilde{\psi}(x-l)\right\rangle = \delta_{k,l} \qquad \forall k,l\in\mathbb{Z}. \tag{1.5}$$

- $$\left\langle \psi(x-k), \tilde{\phi}(x-l)\right\rangle = \left\langle \tilde{\psi}(x-k), \phi(x-l)\right\rangle = \delta_{k,l} \qquad \forall k,l\in\mathbb{Z}. \tag{1.6}$$

- *For any $j\in\mathbb{Z}$ the set $\{\psi_{j,k}, k\in\mathbb{Z}\}$ is a Riesz basis of $V_j$ and for any $j\in\mathbb{Z}$ the set $\left\{\tilde{\psi}_{j,k}, k\in\mathbb{Z}\right\}$ is a Riesz basis of $\tilde{V}_j$.*

- *If moreover $\phi(x)$ and $\tilde{\phi}(x)$ for some $C>0$ and $\forall\xi\in\mathbb{R}$ satisfy*

$$\left|\hat{\phi}(\xi)\right| \le C(1+|\xi|)^{-1} \qquad \textit{and} \qquad \left|\hat{\tilde{\phi}}(\xi)\right| \le C(1+|\xi|)^{-1},$$

*then $\{\psi_{j,k}\}_{j,k\in\mathbb{Z}}$ and $\left\{\tilde{\psi}_{j,k}\right\}_{j,k\in\mathbb{Z}}$ are Riesz bases of $L^2(\mathbb{R})$.*

For the proof of this theorem which relies on techniques based on the Fourier transform, we refer to [42]. The consequence of (1.5) and (1.6) is that the spaces $W_j$ and $\tilde{W}_l$ are orthogonal for all $j \neq l$, the space $W_j$ is orthogonal to $\tilde{V}_l$ for all $l \le j$ and the space $\tilde{W}_j$ is orthogonal to $V_l$ for all $l \le j$. Moreover $W_j$ complements $V_j$ in $V_{j+1}$ and similarly $\tilde{W}_j$ complements $\tilde{V}_j$ in $\tilde{V}_{j+1}$. Then the space $V_j$ can be decomposed:

$$V_j = V_{j_0} \oplus W_{j_0} \oplus W_{j_0+1} \ldots \oplus W_{j-1}$$

and due to (1.5) any function $f \in V_j$ can be expanded into

$$f = \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\phi}_{j,k} \right\rangle \phi_{j,k} = \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\phi}_{j_0,k} \right\rangle \phi_{j_0,k} + \sum_{j=j_0}^{j-1} \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k}. \tag{1.7}$$

The first part of the expansion (1.7) is called a *singlescale representation* of the function $f$ while the second part of the expansion is called a *multiresolution* or *multiscale representation* of the function $f$.
Many properties of wavelets can be formulated by equivalent or necessary conditions on its Fourier transform, on its symbols and its scaling coefficients. For instance, necessary conditions on symbols and scaling coefficient, which are useful for the construction of the dual scaling function or for the computation of scaling coefficients of orthonormal wavelets, are given in the following theorem proved in [18].

**Theorem 10.** *If scaling functions $\phi$ and $\tilde{\phi}$ are mutually biorthogonal then*

*1) the scaling coefficients $h_n$ and $\tilde{h}_n$ satisfy*

$$\sum_{n\in\mathbb{Z}} h_n \tilde{h}_{n-2k} = 2\delta_{0,k} \qquad \forall k \in \mathbb{Z}$$

*2) and the symbols of scaling functions $m(\omega)$ and $\tilde{m}(\omega)$ satisfy*

$$m(\omega)\overline{\tilde{m}(\omega)} + m(\omega+\pi)\overline{\tilde{m}(\omega+\pi)} = 1 \qquad \forall \omega \in \mathbb{R},$$

*where*

$$m(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} h_n \mathrm{e}^{-\mathrm{i}n\omega}, \qquad \tilde{m}(\omega) = \frac{1}{2}\sum_{n\in\mathbb{Z}} \tilde{h}_n \mathrm{e}^{-\mathrm{i}n\omega} \qquad \forall \omega \in \mathbb{R}. \tag{1.8}$$

*Moreover, the conditions 1) and 2) are equivalent. Furthermore, if wavelet coefficients are defined by (1.3) then*

$$\sum_{n\in\mathbb{Z}} g_n \tilde{g}_{n-2k} = 2\delta_{0,k} \qquad \forall k \in \mathbb{Z}$$

*and*

$$\sum_{n\in\mathbb{Z}} h_n \tilde{g}_{n-2k} = \sum_{n\in\mathbb{Z}} g_n \tilde{h}_{n-2k} = 0 \qquad \forall k \in \mathbb{Z}.$$

## 1.4 Oblique projections

A requirement on biorthogonal scaling functions to be also in $V_0$ leads in many cases to globally supported biorthogonal scaling functions [18]. It causes some difficulties such as a complicated evaluation of scalar products with them. Therefore we do not require that

$V_0 = \tilde{V}_0$ and we define so called oblique projections $P_j : L^2(\mathbb{R}) \to V_j$ and $\tilde{P}_j : L^2(\mathbb{R}) \to \tilde{V}_j$ by

$$P_j f = \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\phi}_{j,k} \right\rangle \phi_{j,k}, \qquad \tilde{P}_j f = \sum_{k\in\mathbb{Z}} \left\langle f, \phi_{j,k} \right\rangle \tilde{\phi}_{j,k},$$

and detail operators $Q_j : L^2(\mathbb{R}) \to W_j$, $\tilde{Q}_j : L^2(\mathbb{R}) \to \tilde{W}_j$ by

$$Q_j f = P_{j+1} f - P_j f, \qquad \tilde{Q}_j f = \tilde{P}_{j+1} f - \tilde{P}_j f.$$

Since the spaces $V_j$ are nested, we have $P_j P_l = P_j$ for all $j < l$. Consequently the detail operator $Q_j$ is also a projection on a space $W_j$ and can be expanded into [18]

$$Q_j f = \sum_{k\in\mathbb{Z}} \left\langle f, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k}.$$

Then the space $W_j$ can be also defined as the kernel of $P_j$ in $V_{j+1}$.
It is known from [33] that if $\phi$ is a compactly supported $L^2$-stable refinable function then there always exists a dual scaling function to $\tilde{\phi}$ which is also compactly supported. In the rest of this chapter, we will assume that both scaling functions $\phi$ and $\tilde{\phi}$ are compactly supported. Basic properties of projections $P_j$ and $\tilde{P}$ are described in the next theorem.

**Theorem 11.** *If both scaling functions $\phi$ and $\tilde{\phi}$ are compactly supported and mutually biorthogonal then*

- *The scaling functions $\phi$ and $\tilde{\phi}$ are $L^2$-stable.*
- *Oblique projectors $P_j$ and $\tilde{P}_j$ are $L^2$-bounded independently of $j$.*
- $$\lim_{j\to\infty} \|P_j f - f\| \to 0 \qquad \Longleftrightarrow \qquad \int_{\mathbb{R}} \tilde{\phi}(x)\,dx \sum_{k\in\mathbb{Z}} \phi(x-k) = 1 \quad \textit{a.e.}$$

  *and*

  $$\lim_{j\to\infty} \left\|\tilde{P}_j f - f\right\| \to 0 \qquad \Longleftrightarrow \qquad \int_{\mathbb{R}} \phi(x)\,dx \sum_{k\in\mathbb{Z}} \tilde{\phi}(x-k) = 1 \quad \textit{a.e.}$$

- *Both scaling functions $\phi$ and $\tilde{\phi}$ have non-zero integral, and satisfy*

  $$\int_{\mathbb{R}} \phi(x)\,dx \int_{\mathbb{R}} \tilde{\phi}(x)\,dx = 1.$$

  *Up to a renormalization, we can assume that $\int_{\mathbb{R}} \phi(x)\,dx = \int_{\mathbb{R}} \tilde{\phi}(x)\,dx = 1$. And the corresponding scaling coefficients $h_k$ and $\tilde{h}_k$ satisfy*

  $$\sum_{k\in\mathbb{Z}} h_k = \sum_{k\in\mathbb{Z}} \tilde{h}_k = 2, \qquad \sum_{k\in\mathbb{Z}} (-1)^k h_k = \sum_{k\in\mathbb{Z}} (-1)^k \tilde{h}_k = 0.$$

The proof of Theorem 11 can be found in [18]. It follows from Theorem 11, that for any $f \in L^2(\mathbb{R})$ holds

$$f = \lim_{j\to\infty}\left(\sum_{k\in\mathbb{Z}}\left\langle f, \tilde{\phi}_{j,k}\right\rangle \phi_{j,k}\right) = \lim_{j\to\infty}\left(\sum_{k\in\mathbb{Z}}\left\langle f, \tilde{\phi}_{j_0,k}\right\rangle \phi_{j_0,k} + \sum_{j=j_0}^{j-1}\sum_{k\in\mathbb{Z}}\left\langle f, \tilde{\psi}_{j,k}\right\rangle \psi_{j,k}\right).$$

## 1.5 The Discrete Wavelet Transform

For a computation with wavelets it is in the most cases more advantageous to work with a multiscale representation but in some case it is more efficient to work with a single scale representation (for example evaluation of scalar products with wavelets). Therefore we need an efficient tool which enables to change both representations easily. This tool is called the discrete wavelet transform (DWT). Its first part can be derived from the scaling equation (1.2) and the wavelet equation (1.4). We have from (1.2)

$$\begin{aligned}\tilde{\phi}_{j,k}(x) &= 2^{j/2}\tilde{\phi}\left(2^j x - k\right) = 2^{j/2}\sum_{n\in\mathbb{Z}}\tilde{h}_n\tilde{\phi}\left(2^{j+1}x - 2k - n\right)\\ &= 2^{-1/2}\sum_{m\in\mathbb{Z}}\tilde{h}_{m-2k}2^{(j+1)/2}\tilde{\phi}\left(2^{j+1}x - m\right) = 2^{-1/2}\sum_{m\in\mathbb{Z}}\tilde{h}_{m-2k}\tilde{\phi}_{j+1,m}(x).\end{aligned}$$

This implies that

$$c_{j,k} = 2^{-1/2}\sum_{m\in\mathbb{Z}}\tilde{h}_{m-2k}c_{j+1,m}. \tag{1.9}$$

From (1.4) we obtain

$$\begin{aligned}\tilde{\psi}_{j,k}(x) &= 2^{j/2}\tilde{\psi}\left(2^j x - k\right) = 2^{j/2}\sum_{n\in\mathbb{Z}}\tilde{g}_n\tilde{\phi}\left(2^{j+1}x - 2k - n\right)\\ &= 2^{-1/2}\sum_{m\in\mathbb{Z}}\tilde{g}_{m-2k}2^{(j+1)/2}\tilde{\phi}\left(2^{j+1}x - m\right) = 2^{-1/2}\sum_{m\in\mathbb{Z}}\tilde{g}_{m-2k}\tilde{\phi}_{j+1,m}.\end{aligned}$$

It follows that

$$d_{j,k} = 2^{-1/2}\sum_{m\in\mathbb{Z}}\tilde{g}_{m-2k}c_{j+1,m}. \tag{1.10}$$

The equations (1.9) and (1.10) represent the *decomposition algorithm*. We can also reconstruct coefficients $c_{j+1,k}$ from coefficients $c_{j,k}$ and $d_{j,k}$. From the relation $\tilde{V}_j = \tilde{V}_{j-1} + \tilde{W}_{j-1}$, it follows that

$$\begin{aligned}\sum_{k\in\mathbb{Z}}c_{j,k}\tilde{\phi}_{j,k} &= \sum_{k\in\mathbb{Z}}c_{j-1,k}\tilde{\phi}_{j-1,k} + \sum_{k\in\mathbb{Z}}d_{j-1,k}\tilde{\psi}_{j-1,k}\\ &= 2^{-1/2}\sum_{k\in\mathbb{Z}}c_{j-1,k}\sum_{n\in\mathbb{Z}}\tilde{h}_{n-2k}\tilde{\phi}_{j,k} + 2^{-1/2}\sum_{k\in\mathbb{Z}}d_{j-1,k}\sum_{n\in\mathbb{Z}}\tilde{g}_{n-2k}\tilde{\phi}_{j,k}.\end{aligned}$$

By matching coefficients, we obtain the *reconstruction algorithm*:

$$c_{j,k} = 2^{-1/2} \sum_{n\in\mathbb{Z}} \tilde{h}_{k-2n} c_{j-1,n} + 2^{-1/2} \sum_{n\in\mathbb{Z}} \tilde{g}_{k-2n} d_{j-1,n}. \tag{1.11}$$

The reconstruction algorithm is the second half of the discrete wavelet transform. The first part of (1.11) can be also used to obtain an approximation (prediction) of the coefficients $c_{j,k}$ from the data at coarser scale $j-1$. In practice, we deal with functions with compact support. Then there exist $k_1 \in \mathbb{Z}$, $n \in \mathbb{N}$ such that $c_{j,k} = 0$ for $k > k_1$ and $k \leq k_1 + n$. In this case the discrete wavelet transform can be performed in $\mathcal{O}(n)$ operations.

## 1.6 Polynomial exactness

The rate of decay of the approximation error of a function $f$ defined by $\|P_j f - f\|$ is given by the polynomial exactness of the primal scaling basis and by the regularity of $f$. In the next theorem, equivalent conditions for the polynomial exactness are given.

**Theorem 12.** *Let $\phi$, $\tilde{\phi} \in L^1(\mathbb{R})$ be a compactly supported functions satisfying $\int_{\mathbb{R}} \phi(x)\, dx = 1$. Then the following properties are equivalent:*

- *$\phi$ satisfies the Strang-Fix conditions of order $L-1$, i.e.*

$$\left(\frac{\partial}{\partial\omega}\right)^q \widehat{\phi}(2\pi n) = 0, \qquad n \in \mathbb{Z} \setminus \{0\}, \quad \forall q = 0, \ldots, L-1.$$

- *For all $q = 0, \ldots, L-1$, we can expand the polynomial $x^q$ according to*

$$x^q = \sum_{k\in\mathbb{Z}} \left\langle x^q, \tilde{\phi}\,(x-k) \right\rangle \phi\,(x-k)\,, \qquad a.e.$$

- *The symbol of $\phi$ defined by (1.8) has the factorized form*

$$m\,(\omega) = \left(\frac{1+\mathrm{e}^{-\mathrm{i}\omega}}{2}\right)^L p\,(\omega)\,,$$

  *where $p\,(\omega)$ is a trigonometric polynomial.*

- *The dual wavelet $\tilde{\psi}$ has $L$ vanishing moments, i.e.*

$$\int_{\mathbb{R}} x^q \tilde{\psi}\,(x)\, dx = 0 \qquad \forall q = 0, \ldots, L-1.$$

- *There exists a constant $C > 0$ such that we have for any $f \in H^L(\mathbb{R})$:*

$$\|f - P_j f\|_{H^q(\mathbb{R})} \leq C 2^{-j(L-q)} \, |f|_{H^L(\mathbb{R})}\,, \qquad \forall q = 0, \ldots, L-1.$$

The proof can be found in [18]. Another important condition based on the regularity of the scaling function can be found also in [18]:

**Theorem 13.** *If $\phi$ is a $L^2$-stable compactly supported refinable function in $H^L(\mathbb{R})$ for $L \in \mathbb{N}_0$, then it satisfies the Strang-Fix conditions of order $L$.*

For the analysis of the regularity of the scaling function based on the explicitly know scaling coefficients, we refer for example [38].

## 1.7 B-Splines

In this part we shortly introduce basic properties of B-splines which are frequently used as primal scaling functions.

**Definition 14.** The *B-spline* $B_N$ of degree $N$ is defined by $B_0(x) = \phi_H(x)$ (the Haar scaling function) and then recursively by the convolution:

$$B_N(x) = B_0(x) * B_{N-1}(x) := \int_{\mathbb{R}} B_0(t)\, B_{N-1}(x-t)\, dt, \qquad N \in \mathbb{N}.$$

The following theorem summarizes properties of B-splines.

**Theorem 15.** *For $N \in \mathbb{N}$ the functions $B_N$ have the following properties:*

- *$B_N$ is supported in $[0, N+1]$.*
- $B_N(x) > 0 \qquad \forall x \in (0, N+1)$.
- *The function $B_N$ is symmetric with respect to the point $\frac{N+1}{2}$, i.e.*

$$B_N\left(\frac{N+1}{2} - x\right) = B_N\left(\frac{N+1}{2} + x\right) \qquad \forall x \in \mathbb{R}.$$

- $\int_{\mathbb{R}} B_N(x)\, dx = 1$.
- 

$$B_N(x) = \frac{1}{N!} \sum_{k=0}^{N+1} (-1)^k \binom{N+1}{k} (x-k)_+^N \qquad \forall x \in \mathbb{R},$$

*where $x_+^N = (max\{0, x\})^N$.*

- *The set $\{B_N(2^j x - k)\}_{k \in \mathbb{Z}}$ generates the multiresolution spaces*

$$V_j = \left\{ f \in L^2(\mathbb{R}) \cap C^{N-1}(\mathbb{R}) : f|_{\left[\frac{k}{2^j}, \frac{k+1}{2^j}\right]} \in \Pi_N,\ \forall k \in \mathbb{Z} \right\}.$$

- *$B_N$ is $L^2$-stable.*

- *$B_N$ is a refinable function, i.e. it satisfies (1.2), and nonzero scaling coefficients are given by*

$$h_n = 2^{-N}\binom{N+1}{n} \quad \forall n = 0, \ldots N.$$

The proof of statements of this theorem and other interesting properties of splines can be found in [4, 16, 18, 43]. Now, we can define the primal scaling function as $\phi_N := B_N$, this function reproduces polynomials up to degree $N$. It has been shown in [21] that for each $N$ and any $\tilde{N} \in \mathbb{N}$, $\tilde{N} \geq N$, such that $N + \tilde{N}$ is even, there exists a compactly supported dual scaling function, which is exact of order $\tilde{N}$.

# Chapter 2

# Wavelets on the Bounded Interval

Wavelets on the real line are not usually suitable in applications which are defined on bounded domains. Therefore it is necessary to adapt them first on the bounded interval. The main idea is to retain most of the inner functions, i.e. the scaling functions and wavelets whose supports is contained in the interval, and to treat boundary scaling functions and wavelets separately. In some cases it is possible to take restrictions of some of the overlapping functions but in the most cases it is necessary to construct so called boundary functions. During their construction the important properties of wavelets should be preserved such as a Riesz basis property, a smoothness, a local support of basis functions and a polynomial exactness of the wavelet basis. The main disadvantage of some existing constructions is a large condition number of wavelet bases resulting in a bad numerical stability and bad spectral properties of the corresponding stiffness matrices when solving differential equations numerically. This chapter provides an introduction to wavelets on the bounded interval and unlike the previous chapter we consider here wavelet systems generated by many wavelets and by many scaling functions. Wavelets can be even different at different decomposition levels. All these facts complicate not only notation but also a theory.

## 2.1 Wavelet Basis

We start with a definition of a wavelet basis. We consider here families $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} \subset L^2(0,1)$ of functions where $\mathcal{J}$ is an infinite index set and $\mathcal{J} = \mathcal{J}_\Phi \cup \mathcal{J}_\Psi$, where $\mathcal{J}_\Phi$ is a finite set representing scaling functions living on the coarsest scale. Any index $\lambda \in \mathcal{J}$ is of the form $\lambda = (j,k)$, where $|\lambda| = j$ denotes a scale and $k$ denotes spatial location. The above notation enables us to write wavelet expansions as

$$\mathbf{d}^T\Psi := \sum_{\lambda \in \mathcal{J}} d_\lambda \psi_\lambda.$$

Further, we will use the following shorthand notations for two collections of functions $\Psi$, $\tilde{\Psi} \in L^2(0,1)$:

$$\left\langle \Psi, \tilde{\Psi} \right\rangle_{L^2(0,1)} := \left( \left\langle \psi, \tilde{\psi} \right\rangle_{L^2(0,1)} \right)_{\psi \in \Psi, \tilde{\psi} \in, \tilde{\Psi}}.$$

Thus, the biorthogonality condition can be written as

$$\left\langle \Psi, \tilde{\Psi} \right\rangle = \mathbf{I}.$$

**Definition 16.** The family $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} \subset L^2(0,1)$ is called a *wavelet basis* of $H^s$ for some $\gamma, \widetilde{\gamma} > 0$ and $s \in (-\widetilde{\gamma}, \gamma)$, if

- $\Psi$ normalized in $H^s$ is a Riesz basis of $H^s$; it means that $\Psi$ forms a basis of $H^s$ and there exist constants $c_s, C_s > 0$ such that for all $\mathbf{b} = \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J})$ holds

  $$c_s \|\mathbf{b}\|_{l^2(\mathcal{J})} \leq \left\| \sum_{\lambda \in \mathcal{J}} \frac{b_\lambda \psi_\lambda}{\|\psi_\lambda\|_{H^s}} \right\|_{H^s} \leq C_s \|\mathbf{b}\|_{l^2(\mathcal{J})},$$

  where $\sup c_s$, $\inf C_s$ are called Riesz bounds and $\operatorname{cond}(\Psi) := \dfrac{\inf C_s}{\sup c_s}$ is called the condition number of $\Psi$.

- The functions are local in the sense that $\operatorname{diam}(\operatorname{supp} \psi_\lambda) \lesssim 2^{-|\lambda|} \quad \forall \lambda \in \mathcal{J}$.

- Functions $\psi_\lambda$, $\lambda \in \mathcal{J}_\Psi$, have cancellation properties of order $m$, i.e.,

  $$\left| \int_0^1 v(x)\, \psi_\lambda(x)\, dx \right| \lesssim 2^{-m|\lambda|} |v|_{H^m(0,1)}, \qquad \forall v \in H^m(0,1).$$

  It means that integration against wavelets eliminates smooth parts of functions. It is equivalent with vanishing wavelet moments of order $m$ and with the polynomial exactness of dual multiresolution analysis of order $m-1$.

The wavelet system $\Psi$ is usually constructed with the assistance of a multiresolution analysis.

**Definition 17.** A sequence $\mathcal{V} = \{V_j\}_{j \in \mathbb{N}_{j_0}}$ of closed linear subspaces $V_j \subset H^s$ is called a *multiresolution* or *multiscale analysis*, if the subspaces are nested, i.e.,

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots \subset H^s$$

and is dense in $H$, i.e.

$$\overline{\cup_{j \in \mathbb{N}_{j_0}} V_j}^{H^s} = H^s.$$

We now assume that $V_j$ is spanned by set of scaling functions

$$\Phi_j := \{\phi_{j,k}, k \in \mathcal{I}_j\},$$

where $\mathcal{I}_j$ is a finite index set. Furthermore, the collections $\Phi_j$ will always be assumed to be uniformly stable, uniformly bounded and *uniformly local* in the sense that $\forall k \in \mathcal{I}_j$ and $\forall x \in (0,1)$

$$\operatorname{diam}(\operatorname{supp}\phi_{j,k}) \lesssim 2^{-j} \quad \text{and} \quad \#\left\{k \in \mathcal{I}_j, B(x, 2^{-j}) \cap \operatorname{supp}\phi_{j,k} \neq \emptyset\right\} \lesssim 1,$$

where $B(x, 2^{-j})$ is the ball with radius $2^{-j}$ centered at $x$.
The nestedness of $\mathcal{V}$ and the uniform stability of the Riesz bases imply the existence of a bounded linear operator $\mathbf{M}_{j,0} = \left(m_{l,k}^{j,0}\right)_{l\in\mathcal{I}_{j+1}, k\in\mathcal{I}_j}$ such that

$$\phi_{j,k} = \sum_{l\in\mathcal{I}_{j+1}} m_{l,k}^{j,0}\phi_{j+1,l}.$$

Viewing $\Phi_j$ as a column vector, above refinement relations can be expressed in a matrix form as

$$\Phi_j = \mathbf{M}_{j,0}^T\Phi_{j+1}. \tag{2.1}$$

As a consequence of uniform locality, the matrices $\mathbf{M}_{j,0}$ are *uniformly sparse* i.e. the number of entries per each row and column is uniformly bounded. Similarly as in the previous chapter, the nestedness of $\mathcal{V}$ further implies the existence of the complement spaces $W_j$. Let

$$\Psi_j := \{\psi_{j,k}, k \in \mathcal{J}_j\}, \quad \mathcal{J}_j := \mathcal{I}_{j+1} \setminus \mathcal{I}_j, \quad j \geq j_0,$$

be a Riesz basis of $W_j$. Functions in $\Psi_j$ are called *wavelets*. Since $\Psi_j \subset V_{j+1}$ and $\Phi_{j+1}$ forms a Riesz basis of its span, we have a unique representation

$$\psi_{j,k} = \sum_{l\in\mathcal{I}_{j+1}} m_{l,k}^{j,1}\phi_{j+1,l},$$

which can be again expressed in a matrix form as

$$\Psi_j = \mathbf{M}_{j,1}^T\Phi_{j+1}, \tag{2.2}$$

where $\mathbf{M}_{j,1}$ is a bounded linear operator given by $\mathbf{M}_{j,1} = \left(m_{l,k}^{j,1}\right)_{l\in\mathcal{I}_{j+1}, k\in\mathcal{J}_j}$. Further, we assume that collection $\Psi_j$ is uniformly local and then $\mathbf{M}_{j,1}$ is also uniformly sparse. The refinement relations (2.1) and (2.2) lead to refinement equations in a matrix form

$$\begin{pmatrix}\Phi_j\\ \Psi_j\end{pmatrix} = \mathbf{M}_j^T\Phi_{j+1},$$

with a refinement matrix $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$. Matrices $\mathbf{M}_j$ are invertible and let inverse matrices be defined by

$$\mathbf{M}_j^{-1} := \mathbf{G}_j = \begin{pmatrix}\mathbf{G}_{j,0}\\ \mathbf{G}_{j,1}\end{pmatrix}.$$

Inverses of sparse matrices are not in the general case sparse. However, when we require uniformly local dual wavelets then the inverses of these matrices have to be uniformly sparse. In this case, wavelets are usually constructed by a method of stable completion proposed in [6].

**Definition 18.** Any $\mathbf{M}_{j,1} \in [l^2(\mathcal{J}_j), l^2(\mathcal{I}_{j+1})]$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\kappa(\mathbf{M}_j),\ \kappa(\mathbf{M}_j^{-1}) = \mathcal{O}(1), \quad j \to \infty,$$

where $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$.

It is known that $\Phi_j \cup \Psi_j$ is uniformly stable if and only if $\mathbf{M}_{j,1}$ is a stable completion of $\mathbf{M}_{j,0}$, see [6]. However, it does not imply the Riesz stability over all levels.

## 2.2 Multiscale Transform

The multiscale basis of $V_J$ is given by

$$\Psi^J = \Phi_{j_0} \cup \bigcup_{j=j_0}^{J-1} \Psi_j. \tag{2.3}$$

Since the union of subspaces $V_j$ is dense in $H^s$, a multiscale basis of $H^s$ is given by

$$\Psi = \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j,$$

and we can split $\mathcal{J}$ into two index sets

$$\mathcal{J}_\phi := \{(j_0 - 1, k), k \in \mathcal{I}_j\}, \quad \mathcal{J}_\psi := \{(j,k), j \geq j_0, k \in \mathcal{J}_j\}.$$

From (2.3) it follows that any $v \in V_J$ has a *single-scale representation*

$$v = \mathbf{c}_J^T \Phi = \sum_{k \in \mathcal{I}_j} c_{j,k} \phi_{j,k},$$

as well as a *multiscale representation*

$$v = \mathbf{c}_{j_0}^T \Phi_{j_0} + \mathbf{d}_{j_0}^T \Psi_{j_0} + \ldots + \mathbf{d}_{J-1}^T \Psi_{J-1} = \sum_{k \in I_{j_0}} c_{j_0,k} \phi_{j_0,k} + \sum_{j=j_0}^{J-1} \sum_{k \in J_j} d_{j,k} \psi_{j,k}.$$

The corresponding vectors of the single-scale and multiscale representations are related by the *multiscale transformation* $\mathbf{T}_J : l^2(\mathcal{I}_J) \to l^2(\mathcal{I}_J)$:

$$\mathbf{c}_J = \mathbf{T}_J \left(\mathbf{c}_{j_0}^T, \mathbf{d}_{j_0}^T, \ldots \mathbf{d}_{J-1}^T\right)^T.$$

From refinement relations (2.1) and (2.2), it follows that

$$\mathbf{c}_j^T \Phi_j + \mathbf{d}_j^T \Psi_j = \left(\mathbf{M}_{j,0}\mathbf{c}_j + \mathbf{M}_{j,1}\mathbf{d}_j\right)^T \Phi_{j+1} = \mathbf{c}_{j+1}^T \Phi_{j+1}$$

and then the multiscale transform $\mathbf{T}_J$ is given by:

$$\mathbf{T}_J = \mathbf{T}_{J,J-1} \dots \mathbf{T}_{J,j_0}, \quad \text{where} \quad \mathbf{T}_{J,j} = \begin{pmatrix} \mathbf{M}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

To determine the inverse multiscale transform $\mathbf{T}_J^{-1}$, note that

$$\mathbf{c}_{j+1}^T \Phi_{j+1} = \mathbf{G}_{j,0}^T \mathbf{c}_{j+1}^T \Phi_j + \mathbf{G}_{j,1}^T \mathbf{c}_{j+1}^T \Psi_j = \mathbf{c}_j^T \Phi_j + \mathbf{d}_j^T \Psi_j.$$

Thus, the inverse multiscale transform $\mathbf{T}_J^{-1}$ can be obtained by applying inverses of the matrices $\mathbf{T}_{J,j}$ in the opposite order:

$$\mathbf{T}_J^{-1} = \mathbf{T}_{J,j_0}^{-1} \dots \mathbf{T}_{J,J-1}^{-1}, \quad \text{where} \quad \mathbf{T}_{J,j}^{-1} = \begin{pmatrix} \mathbf{G}_j & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix}.$$

If refinement matrices $\mathbf{M}_j$ and $\mathbf{G}_j$ are uniformly sparse then both the multiscale transform $\mathbf{T}_J$ and the inverse multiscale transform $\mathbf{T}_J^{-1}$ can be performed in $\mathcal{O}(N_J)$ operations, where $N_j$ is the dimension of the space $V_j$. The next theorem shows a relation between properties of the multiscale basis $\Psi$ and the multiscale transform $\mathbf{T}_J$.

**Theorem 19.** *Assume that $\Phi_j$ are uniformly stable. Then $\mathbf{T}_J$ are well-conditioned or stable in the sense of $\kappa(\mathbf{T}_J)$, $\kappa(\mathbf{T}_J^{-1}) = \mathcal{O}(1)$ if and only if $\Psi$ is a Riesz basis in a Hilbert space $H$.*

For further details, we refer to [23].

## 2.3 Riesz Bases in Sobolev Spaces

As was already mentioned in the previous chapter, any wavelet compression algorithm based on removing small coefficients can be reasonable only when wavelets form a Riesz basis. The following theorem from [40] gives useful characterization of Riesz bases.

**Theorem 20.** *Let $j_0$ be the coarsest level and let*

$$V_{j_0} \subset V_{j_0+1} \subset \dots \subset H, \qquad \widetilde{V}_{j_0} \subset \widetilde{V}_{j_0+1} \subset \dots \subset H$$

*be sequences of closed subspaces of $H$ such that with $\dim V_j = \dim \widetilde{V}_j$, then the following statements are equivalent:*

- *There exist uniform Riesz bases $\Phi_j$ and $\widetilde{\Phi}_j$ for $V_j$ and $\widetilde{V}_j$ such that $\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle$ is invertible and the inverses are uniformly bounded.*

- $$\inf_{j\in\mathbb{N}_0} \inf_{0\neq\tilde{v}\in\tilde{V}_{j_0+j}} \sup_{0\neq v\in V_{j_0+j}} \frac{|\langle\tilde{v},v\rangle|}{||\tilde{v}||\,||v||} > 0.$$

- *There exist unique uniformly bounded projections $P_j : H \to V_j$ with* $\mathrm{Im}(I-P_j) = \tilde{V}_j^{\perp}$ *and these projections are given by*
$$P_j x = \left\langle x, \widetilde{\Phi}_j \right\rangle \left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle^{-1} \Phi_j.$$

- *To any uniform Riesz basis for $V_j$ there exist a unique uniform biorthogonal Riesz basis in $\widetilde{V}_j$.*

*Let any of the above conditions be satisfied and moreover let the following minimum angle condition hold*

$$\sup_{j\in\mathbb{N}_0} \cos\angle(V_{j_0+j}, W_{j_0+j}) < 1 \quad \textit{where} \quad \cos\angle(V_j, W_j) := \sup_{0\neq v\in V_j,\, 0\neq w\in W_j} \frac{|\langle w,v\rangle|}{||w||\,||v||},$$

*then $(I-P_j)|_{W_j} : W_j \to V_{j+1}\cap\widetilde{V}_j^{\perp}$ is invertible and the inverses are uniformly bounded.*

The first part of the previous theorem enables to formulate prior results from [25] without explicit knowledge of some biorthogonal bases while the second part was used in [40] to a construction of biorthogonal wavelets on non-uniform meshes. In this construction both primal and dual wavelets are known in explicit form, have a compact support and are piecewise polynomials. The following two theorems state how Riesz bases for a range of Sobolev spaces can be created. The first theorem describes the case, when we have two mutually biorthogonal bases, while the second one describes the case, when a dual biorthogonal basis in not known.

**Theorem 21.** *Let $j_0$ be the coarsest level and let*

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset L^2(0,1), \qquad \widetilde{V}_{j_0} \subset \widetilde{V}_{j_0+1} \subset \ldots \subset L^2(0,1)$$

*be sequences of primal and dual spaces which are mutually biorthogonal and which are equipped uniform $L^2(0,1)-$Riesz bases $\Phi_j$ and $\widetilde{\Phi}_j$ for $V_j$ and $\widetilde{V}_j$, respectively. In addition, for some $0<\gamma<d$, let*

$$\inf_{v_j\in V_j} ||v-v_j||_{L^2(0,1)} \lesssim 2^{-jd}||v||_{H^d(0,1)} \qquad \forall v\in H^d(0,1),$$

*(Jackson or direct estimate) and*

$$||v_j||_{H^s(0,1)} \lesssim 2^{js}||v_j||_{L^2(0,1)} \qquad \forall v_j\in V_j,\ s\in[0,\gamma),$$

*(Bernstein or inverse estimate) and let similar estimates be valid at the dual side with $V_j, d, \gamma, H^s(0,1)$ reading as $\widetilde{V}_j, \widetilde{d}, \widetilde{\gamma}, \widetilde{H}^s(0,1)$. And let $\Psi_j$ be uniform $L^2(0,1)-$Riesz bases for $W_j := V_{j+1}\cap\widetilde{V}_j^{\perp L^2(0,1)}$, then for $s\in(-\widetilde{\gamma},\gamma)$ the collection*

$$\Phi_{j_0} \cup \bigcup_{j\in\mathbb{N}_0} 2^{-sj}\Psi_{j_0+j}$$

*is a Riesz basis for $H^s(0,1)$, where $H^s(0,1) := (H^{-s}(0,1))'$ for $s<0$.*

This theorem is a consequence of results from [25, 40]. The following theorem from [29] summarizes results from [23, 25]:

**Theorem 22.** *Let $j_0$ be the coarsest level and let*

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset L^2(0,1), \qquad \widetilde{V}_{j_0} \subset \widetilde{V}_{j_0+1} \subset \ldots \subset L^2(0,1)$$

*be sequences of primal and dual spaces with*

$$\dim V_j = \dim \widetilde{V}_j$$

*such that for uniform $L^2(0,1)-$Riesz bases $\Phi_j$ and $\widetilde{\Phi}_j$ for $V_j$ and $\widetilde{V}_j$, respectively,*

$$\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L^2(0,1)}^{-1}$$

*exists with a uniformly bounded spectral norm. In addition, for some $0 < \gamma < d$, let*

$$\inf_{v_j \in V_j} ||v - v_j||_{L^2(0,1)} \lesssim 2^{-jd} ||v||_{\mathcal{H}^d(0,1)} \qquad \forall v \in \mathcal{H}^d(0,1),$$

*(Jackson or direct estimate) and (Bernstein or inverse estimate)*

$$||v_j||_{\mathcal{H}^s(0,1)} \lesssim 2^{js} ||v_j||_{L^2(0,1)} \qquad \forall v_j \in V_j, \;\; s \in [0, \gamma),$$

*where, for $s \in [0, d]$, $\mathcal{H}^s(0,1) = \left[L^2(0,1), H^d(0,1) \cap H_0^1(0,1)\right]_{s/d}$, and let similar estimates be valid at the dual side with $V_j, d, \gamma, \mathcal{H}^s(0,1)$ reading as $\widetilde{V}_j, \widetilde{d}, \widetilde{\gamma}, \widetilde{\mathcal{H}}^s(0,1)$. And let $\Psi_j$ be uniform $L^2(0,1)-$Riesz bases for $W_j := V_{j+1} \cap \widetilde{V}_j^{\perp L^2(0,1)}$, then for $s \in (-\widetilde{\gamma}, \gamma)$ the collection*

$$\Phi_{j_0} \cup \bigcup_{j \in \mathbb{N}_0} 2^{-sj} \Psi_{j_0+j}$$

*is a Riesz basis for $\mathcal{H}^s(0,1)$, where $\mathcal{H}^s(0,1) := (\mathcal{H}^{-s}(0,1))'$ for $s < 0$.*

Concerning validity of direct and inverse estimates, it is well-known [17] that a direct estimate of order $d$ is satisfied when all polynomials of order $d$ satisfying possibly boundary conditions are included in the space $V_{j_0}$, while an inverse estimate of order $\gamma$ is known to hold with $\gamma = r + \frac{3}{2}$ when spaces $V_j$ are spanned by piecewise smooth $C^r(0,1)$ functions for some $r \in \{-1, 0, 1, \ldots\}$, where $r = -1$ means that no global continuity is satisfied. Further generalization of previous works was proposed in [36]. It is useful especially in the case when a proposed basis in not a Riesz basis of the space $L^2(0,1)$.

**Theorem 23.** *Let $j_0$ be the coarsest level and let for some $0 < \gamma$,*

$$V_{j_0} \subset V_{j_0+1} \subset V_{j_0+2} \subset \ldots \subset H^s(0,1) \quad \forall s \in [0, \gamma)$$

*be a sequence of primal spaces with uniformly local and uniformly stable bases $\Phi_j$ for $V_j$ which is a Bessel sequence in $H^s(0,1)$, $\forall s \in (0,\gamma)$. In addition, for some $\gamma \leq d$, let*

$$\inf_{v_j \in V_j} ||v - v_j||_{L^2(0,1)} \lesssim 2^{-jd} ||v||_{H^d(0,1)} \qquad \forall v \in H^d(0,1)$$

*and let there exist a projection $P_j : V_{j+1} \to V_j$ and $0 < \mu < \gamma$ such that*

$$||P_m \cdots P_{n-1}|| \lesssim 2^{\mu(n-m)} \qquad \forall m, n \in \mathbb{N} \text{ with } j_0 \leq m < n.$$

*And let $\Psi_j$ be uniform $L^2(0,1)-$Riesz bases for $W_j := \mathrm{Ker} P_j$, then for $s \in (\mu, \gamma)$ the collection*

$$\Phi_{j_0} \cup \bigcup_{j \in \mathbb{N}_0} 2^{-sj} \Psi_{j_0 + j}$$

*is a Riesz basis for $H^s(0,1)$.*

Some sufficient conditions for $\Phi_j$ to be a Bessel sequence are given in [35].

## 2.4 An Application of Riesz Basis Property

We show here that condition numbers of stiffness matrices arising from discretization of elliptic partial differential equations by wavelets depend on Riesz constants of a wavelet basis. Therefore it is necessary construct wavelet bases which are well-conditioned in the sense that their Riesz condition number is as small as possible. We consider here the following Dirichlet problem

$$u - \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = f \quad \text{in} \quad \Omega = (0,1)^d \qquad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega \tag{2.4}$$

for given $f \in H^{-1}(\Omega)$. A Riesz wavelet basis for $H_0^1(\Omega)$ can be constructed by a tensor product of univariate Riesz wavelet bases. Indeed, let $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\}$ be after appropriate normalization a Riesz wavelet basis for spaces $L^2(0,1)$ and $H_0^1(0,1)$ then

$$\mathbf{\Psi} = \left\{ \boldsymbol{\psi}_{\boldsymbol{\lambda}} := \frac{\otimes_{j=1}^d \psi_{\lambda_m}}{\left\| \otimes_{j=1}^d \psi_{\lambda_m} \right\|_{H^1(\Omega)}}, \boldsymbol{\lambda} \in \mathcal{J}^d \right\}$$

is a Riesz basis for $H_0^1(\Omega)$ (see [31]) with the Riesz constants (see [28])

$$\min(c_0, c_1)\, c_0^{d-1} \|\mathbf{b}\|_{l^2(\mathcal{J}^d)}^2 \leq \left\| \sum_{\boldsymbol{\lambda} \in \mathcal{J}^d} b_{\boldsymbol{\lambda}} \boldsymbol{\psi}_{\boldsymbol{\lambda}} \right\|_{H^1(\Omega)}^2 \leq \max(C_0, C_1)\, C_0^{d-1} \|\mathbf{b}\|_{l^2(\mathcal{J}^d)}^2 \tag{2.5}$$

$\forall \mathbf{b} \in l^2(\mathcal{J}^d)$, where constants $c_0$, $C_0$, $c_1$, $C_1$ are Riesz constants with respect to spaces $L^2$ and $H_1$, respectively, and the index set $\mathcal{J}^d$ is defined by $\mathcal{J}^d := \{\boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d),\ \lambda_i \in \mathcal{J}\}$ Writing

$$u = \mathbf{u}^T \mathbf{\Psi} := \sum_{\boldsymbol{\lambda} \in \mathcal{J}^d} \mathbf{u}_{\boldsymbol{\lambda}} \boldsymbol{\psi}_{\boldsymbol{\lambda}} \qquad \text{and} \qquad \mathbf{f} = (f(\boldsymbol{\psi}_{\boldsymbol{\lambda}}))_{\boldsymbol{\lambda} \in \mathcal{J}^d},$$

then an equivalent formulation of (2.4) is

$$\mathbf{A}\mathbf{u} = \mathbf{f}$$

with

$$\mathbf{A} = \mathbf{D}^{-1/2}\left(\mathbf{M} \otimes \ldots \otimes \mathbf{M} + \mathbf{S} \otimes \ldots \otimes \mathbf{M} + \cdots + \mathbf{M} \otimes \ldots \otimes \mathbf{S}\right)\mathbf{D}^{-1/2},$$

where $\mathbf{D} = \text{diag}\ \left[\left\|\otimes_{j=1}^{d}\psi_{\lambda_m}\right\|_{H^1(\Omega)}\right]_{\lambda\in\mathcal{J}^d}$, and

$$\mathbf{S} = \left(\int_0^1 \frac{\partial\psi_\lambda}{\partial x}\frac{\partial\psi_\mu}{\partial x}\ dx\right)_{\lambda,\mu\in\mathcal{J}} \qquad \text{and} \qquad \mathbf{M} = \left(\int_0^1 \psi_\lambda\,\psi_\mu\ dx\right)_{\lambda,\mu\in\mathcal{J}}$$

are the one-dimensional stiffness and the mass matrices, respectively. Then (2.5) implies

$$\text{cond}\,(\mathbf{A}) \leq \frac{\max\left(C_0, C_1\right) C_0^{d-1}}{\min\left(c_0, c_1\right) c_0^{d-1}}.$$

In general case, let us assume, that we have the following variational problem: for given $f \in \mathcal{H}'$ find $u \in \mathcal{H}$ such that

$$a(u,v) = f(v) \qquad \forall v \in \mathcal{H}, \tag{2.6}$$

where $\mathcal{H}$ is a Hilbert space and $a$ is a continuous bilinear form. Then, we define the operator $\mathcal{A} : \mathcal{H} \to \mathcal{H}'$ by

$$\mathcal{A}(u)(v) = a(u,v) \qquad \forall v \in \mathcal{H},$$

and then (2.6) is equivalent to

$$\mathcal{A}(u) = f.$$

If $a$ is $\mathcal{H}-$elliptic, then there exist positive constants $c_{\mathcal{A}}$, $C_{\mathcal{A}}$ such that

$$c_{\mathcal{A}}\left\|v\right\|_{\mathcal{H}} \leq \left\|\mathcal{A}(v)\right\|_{\mathcal{H}'} \leq C_{\mathcal{A}}\left\|v\right\|_{\mathcal{H}} \qquad \forall v \in \mathcal{H}. \tag{2.7}$$

Moreover, we will assume that we have a suitable wavelet basis $\mathbf{\Psi}$ of the space $\mathcal{H}$ normalized in $\mathcal{H}$ with Riesz constants $c$, $C$ and we define $\mathbf{A} = a\left(\mathbf{\Psi}, \mathbf{\Psi}\right)$ and $\mathbf{f} = f\left(\mathbf{\Psi}\right)$, then

$$\mathcal{A}(u) = f \qquad \Longleftrightarrow \qquad \mathbf{A}\mathbf{u} = \mathbf{f},$$

where $u = \mathbf{u}^T\mathbf{\Psi}$, and

$$\text{cond}\,(\mathbf{A}) \leq \frac{C^2 C_{\mathcal{A}}}{c^2 c_{\mathcal{A}}}.$$

Proof can be found in [3]. Thus we can conclude that the condition number of the stiffness matrix $\mathbf{A}$ is bounded which favorably influences a number of iteration needed to solve a system of equations resulting from a wavelet discretization of (2.6). And then it greatly influences efficiency of adaptive wavelet methods. Therefore it is useful to develop well-conditioned wavelet bases on the interval. Well-conditioned wavelet basis for different types of wavelets and for different types of boundary conditions were already constructed in [7, 8, 10, 11, 37].

# Chapter 3

# Selected Results

In the last chapter we present selected results published in the following four papers:

- Černá, D.; Finěk, V.; Najzar, K.: *On the exact values of coefficients of Coiflets*, Cent. Eur. J. Math. **6(1)**, (2008), pp. 159-170.
- Černá, D.; Finěk, V.: *Construction of optimally conditioned cubic spline wavelets on the interval*, Adv. Comput. Math. **34(2)**, (2011), pp. 219-252.
- Černá, D.; Finěk, V.: *Cubic Spline Wavelets with Complementary Boundary Conditions*, Appl. Math. Comput. **219**, (2012), pp. 1853-1865.
- Černá, D.; Finěk, V.: *Wavelet basis of cubic splines on the hypercube satisfying homogeneous boundary conditions*, Int. J. Wavelets Multi. **13(3)**, (2015), pp. 1550014/1-21.
- Černá, D.; Finěk, V.: *On a sparse representation of a $n$-dimensional Laplacian in wavelet coordinates*, Result. Math., DOI 10.1007/s00025-015-0488-5, (2015).

We shortly introduce here these papers and then we include them into this work.

## 3.1 On the Exact Values of Coefficients of Coiflets

In 1989, R. Coifman suggested orthonormal wavelets in $L_2(\mathbb{R})$ with vanishing moments for both scaling and wavelet functions. In practical applications these wavelets are useful due to their nearly linear phase and almost interpolating property. For more details we refer to [34]. They were first constructed by I. Daubechies [26, 27] and she named them coiflets. She created coiflets by setting an equal number $N$ of vanishing wavelet moments and vanishing scaling moments for even $N$ and the length of support $3N$, see [26, 27]. It was noticed in [1] that these coiflets has one additional vanishing scaling moment than imposed. Another types of coiflets can be found in literature. For example C. S. Burrus and J. E. Odegard [5] constructed coiflets with $N$ vanishing moments for odd $N$ and the length of support $3N + 1$ which has two additional vanishing scaling moments. Another approach proposed in [34] consists in a parametrization of coiflets by the first moment of the scaling function. By allowing noninteger values for this parameter, the interpolation and linear phase properties of coiflets can be further optimized.

In the paper "On the Exact Values of Coefficients of Coiflets" [14], we proposed a system of necessary conditions which is redundant free and more simple than other known systems due to elimination of some quadratic (orthonormality) conditions, thus a computation of scaling coefficient of coiflets is substantially simplified and enables to find the exact values of the scaling coefficients up to filters of the length 8 and two further with filters of the length 12. For scaling coefficients of coiflets with filters of the length 14 we obtained two quadratic equations, which can be transformed to polynomial of degree 4 and there is an algebraic formula to solve them. For larger filters up to filters of the length 20, we were able to find all possible solutions by employing a Gröbner basis method. Finally, we verified orthonormality by the sufficient Lawton criterion [32] and found that all solutions correspond to an orthonormal wavelet. Obtained solutions are not of the same quality, because also their smoothness and symmetry plays a role. For this reason, we also computed their Sobolev exponents of smoothness by methods proposed in [30, 41].

There is a number of numerical methods which were used to find scaling coefficients of coiflets but these methods enable to derive only one particular solution for each system and the convergence and the obtained solution depends on the initial starting point, thus it is difficult to find all possible solutions. Moreover, the coefficients for length greater than 16 are given with less precision due to the round-off error [27].

## 3.2 Construction of Optimally Conditioned Cubic Spline Wavelets on the Interval

The first biorthogonal spline wavelet bases on the unit interval were constructed in [24]. In this construction both primal and dual bases functions are compactly supported. However in the most cases, these bases have relatively large condition numbers which causes problems in practical applications. Many modifications improving condition numbers were proposed. We mention only a construction proposed by M. Primbs [37] which seems to outperform the previous constructions with respect to the Riesz bounds as well as spectral properties of the corresponding stiffness matrices in the case of linear and quadratic spline wavelets.

In the paper "Construction of Optimally Conditioned Cubic Spline Wavelets on the Interval" [7], we focused on cubic spline wavelets and we constructed spline wavelet bases on the interval with condition numbers which are close to condition numbers of spline wavelet bases on the real line. In this sense, they are optimally conditioned because it is known that a condition number of the wavelet basis on the interval can not be better then a condition number of a wavelet basis on the real line [2]. From the viewpoint of numerical stability, ideal wavelet bases are orthogonal wavelet bases. However, they are usually avoided in

the numerical treatment of partial differential and integral equations, because they are not usually known in a closed form, sufficiently smooth orthogonal wavelets typically have a large support and it is not possible to increase a number of vanishing wavelet moments independent from the order of accuracy.
Constructed wavelets have the following properties:

- *Riesz basis property.* Functions form a Riesz basis of the space $L^2(0,1)$.
- *Locality.* Basis functions are local.
- *Biorthogonality.* Primal and dual wavelet bases form a biorthogonal pair.
- *Polynomial exactness.* Primal multiresolution analysis has a polynomial exactness of order $N$ and the dual multiresolution analysis has a polynomial exactness of order $\tilde{N}$. As in [21], $N + \tilde{N}$ has to be even and $\tilde{N} \geq N$.
- *Smoothness.* Certain smoothness for primal and dual wavelet basis functions.
- *Closed form.* Primal scaling functions and wavelets are known in the closed form.
- *Well-conditioned bases.* Constructed wavelet bases have improved condition numbers in comparison with previous constructions of the same type.

The primal scaling functions are B-splines, which have been used also in [37]. Then we constructed a dual multiresolution analysis which is generated by three types of scaling functions. Inner scaling functions are the same as in [21] and there are two types of boundary scaling functions. Scaling functions of the first type are defined to preserve the prescribed polynomial exactness in the same way as in [22]. Scaling functions of the second type are constructed to be as similar as possible to restrictions of inner scaling functions. Consequently we computed refinement matrices and constructed wavelets by a method of stable completion. The construction of initial stable completion is along the lines of [24]. Furthermore, we showed that the constructed set of functions are indeed a Riesz basis for the space $L^2(0,1)$ and for the Sobolev space $H^s(0,1)$ for a certain range of $s$. Finally, we adapted primal bases to homogeneous Dirichlet boundary conditions of the first order and we compared quantitative properties of the constructed bases and the efficiency of an adaptive wavelet scheme for several spline wavelet bases to demonstrate a superiority of our construction. Numerical examples were presented for one-dimensional and two-dimensional Poisson equations where the solution has a steep gradient.

## 3.3 Cubic Spline Wavelets with Complementary Boundary Conditions

In the paper "Cubic Spline Wavelets with Complementary Boundary Conditions" [8], we constructed a new stable cubic spline wavelet basis on the interval satisfying complementary boundary conditions of the second order i.e. the primal wavelet basis is adapted to

homogeneous Dirichlet boundary conditions of the second order, while the dual wavelet basis preserves the full degree of polynomial exactness. Primal wavelets have six vanishing moments. Moreover, we proposed further decomposition of the scaling basis at the coarsest level. We decomposed the scaling basis $\Phi_4$ into the scaling basis $\Phi_3$ and the wavelet basis $\Psi_3$. These new wavelets from $\Psi_3$ have four vanishing moments while supports of new boundary scaling functions from $\Phi_3$ overlap in contrast to boundary scaling functions from $\Phi_4$. This modification leads to improved Riesz condition numbers of the proposed basis.
The primal scaling functions are B-splines satisfying homogeneous Dirichlet boundary conditions of the second order. Then in the similar way as in [7], we constructed a dual multiresolution analysis which is generated by three types of scaling functions. Inner scaling functions are the same as in [21] and there are two types of boundary scaling functions. Scaling functions of the first type are defined to preserve the prescribed polynomial exactness while scaling functions of the second type are constructed to be as similar as possible to restrictions of inner scaling functions. Consequently we computed refinement matrices and constructed wavelets by a method of stable completion. We proposed a new construction of the initial stable completion because the standard construction from [24] led to singular matrices. Finally, we presented quantitative properties of the proposed basis and we compared them with some other cubic spline wavelet bases to show superiority of our construction. Numerical examples were presented for the two-dimensional biharmonic equation where the solution has a steep gradient.

## 3.4 Wavelet Basis of Cubic Splines on the Hypercube Satisfying Homogeneous Boundary Conditions

In the paper "Wavelet Basis of Cubic Splines on the Hypercube Satisfying Homogeneous Boundary Conditions" [12], we constructed new cubic spline wavelet basis on the hypercube that is well-conditioned, adapted to homogeneous Dirichlet boundary conditions and the wavelets have two vanishing moments. Proposed wavelets have the same properties as wavelets in the construction [7] with one exception. We do not require compact support for dual functions which enables to construct primal functions with better properties. Dual functions are not in fact used in some applications of wavelets such as numerical solution of linear differential equations. The advantage of our construction in comparison with similar cubic spline wavelets with local dual functions [7, 8, 24, 37] is that the support of wavelets is shorter, Riesz condition numbers are smaller and another advantage is also a simple construction. Then stiffness matrices arising from discretization of elliptic problems using proposed wavelets have uniformly bounded condition numbers and these condition numbers are small. It leads in combination with shorter support to more efficient numerical solvers.
The primal scaling functions are B-splines, which have been used also in [7]. Then we constructed a primal wavelet basis generated by one inner and two boundary wavelets. Inner wavelets are generated by a single function supported in the interval $[0, 5]$ and there

are at each side two boundary wavelets. The first one is supported in the interval $[0, 4]$ and the second one is supported in the interval $[0, 3]$. All three types of wavelet are constructed to be orthogonal to continuous piecewise linear functions which are linear on pieces $\left[\frac{k}{2}, \frac{k+1}{2}\right]$ for $k \in \mathbb{N}$. A space generated by these continuous piecewise linear functions forms a dual multiresolution space which is consequently used in the proof of the Riesz basis property. Moreover from the construction immediately follows that constructed wavelets have two vanishing wavelet moments. Finally, we presented quantitative properties of the constructed basis and we also provided a numerical example to show an efficiency of Galerkin method using constructed basis.

## 3.5 On a Sparse Representation of a $n$-dimensional Laplacian in Wavelet Coordinates

A general concept for solving of operator equations by means of wavelets was proposed by A. Cohen, W. Dahmen and R. DeVore in [19, 20]. It consists of the following steps: transformation of the variational formulation into the well-conditioned infinite-dimensional problem in the space $l^2$, finding of the convergent iteration process for the $l^2-$ problem and finally a derivation of its computable version. The aim is to find an approximation of the unknown solution $u$ which should correspond to the best $N$-term approximation, and the associated computational work should be proportional to the number of unknowns. Essential components to achieve this goal are well-conditioned wavelet stiffness matrices and an efficient approximate multiplication of quasi-sparse wavelet stiffness matrices with vectors.

In [19], authors exploited an off-diagonal decay of entries of the wavelet stiffness matrices and designed a numerical routine **APPLY** which approximates the exact matrix-vector product with the desired tolerance $\epsilon$ and that has linear computational complexity, up to sorting operations. The idea of **APPLY** is following: To truncate $\mathbf{A}$ in scale by zeroing $a_{i,j}$ whenever $\delta(i,j) > k$ ($\delta$ represents the level difference of two functions in the wavelet expansion) and denote resulting matrix by $\mathbf{A_k}$. At the same time to sort vector entries $\mathbf{v}$ with respect to the size of their absolute values. One obtains $\mathbf{v_k}$ by retaining $2^k$ biggest coefficients in absolute values of $\mathbf{v}$ and setting all other equal to zero. The maximum value of $k$ should be determined to reach a desired accuracy of approximation. Then one computes an approximation of $\mathbf{Av}$ by

$$\mathbf{w} := \mathbf{A_k v_0} + \mathbf{A_{k-1}}(\mathbf{v_1} - \mathbf{v_0}) + \ldots + \mathbf{A_0}(\mathbf{v_k} - \mathbf{v_{k-1}}) \qquad (3.1)$$

with the aim to balance both accuracy and computational complexity at the same time. Improvements of this scheme were proposed in [9, 28, 39]. Although the **APPLY** routine has optimal computational complexity, its application is relatively time consuming and moreover it is not easy to implement it efficiently.

In the paper "On a Sparse Representation of a $n$-dimensional Laplacian in Wavelet Coordinates" [13], we constructed a wavelet basis based on Hermite cubic splines with respect to which both the mass matrix and the stiffness matrix corresponding to one dimensional Poisson equation are sparse. This means that the number of nonzero elements in any column is bounded independently of matrix size while stiffness matrices in wavelet coordinates are usually only quasi sparse. Then, matrix-vector multiplication can be performed exactly with linear complexity for any second order PDEs with constant coefficients. Moreover, the proposed basis is very well-conditioned for low decomposition levels. Small condition numbers for low decomposition levels and a sparse structure of stiffness matrices are kept for any second order PDEs with constant coefficients, which are well-conditioned in the sense of (2.7), and moreover they are independent of the space dimension. Wavelets with similar properties were already proposed in [29]. Our wavelets generate the same multiresolution spaces as wavelets from [29] but have improved condition numbers. In comparison with wavelets from [29], we constructed two new wavelets (the first two wavelets are the same) and we also modified boundary scaling functions at the coarsest level as well as wavelets at the coarsest level.

Our construction proceeded in this way. First, we constructed four wavelets in such a way that wavelets from the space $W_{n+1}$ are orthogonal to the scaling functions from the space $V_n$ for $n \geq 1$. This property ensures that both the mass and stiffness matrices corresponding to the one-dimensional Laplacian have at most three wavelet blocks of nonzero elements in any column and, consequently, the number of nonzero elements in any column is bounded independently of matrix size. The first two wavelets have supports in $[-1, 1]$. They are uniquely determined by their orthogonality to cubic polynomials and by imposing that the first one is odd and the second one is even. The other two wavelets have supports in $[-2, 2]$. We impose on them the above orthogonality condition again, which will be ensured by requiring that they are orthogonal to cubic polynomials on intervals $[-2, 0]$ and $[0, 2]$, respectively. Again, the first of them should be odd, and the second one even. There remains several free parameters. To obtain a more sparse stiffness matrix and a better conditioned wavelet basis, we use these free parameters to prescribe the orthogonality of the first derivative of constructed wavelets to the first derivative of the first two wavelets. In the next step, we modified boundary scaling functions at the coarsest level and also wavelets at the coarsest level to further improve condition numbers of the constructed wavelet basis and to preserve or improve a sparse structure of the stiffness matrix corresponding to the one-dimensional Laplacian, and a sparser structure of the mass matrix, respectively. A span of these new functions will be the same as the span of the original functions. In [13], we proved that the constructed basis is a Riesz basis and computed condition numbers for model problems and compared them with condition numbers for a similar wavelet basis proposed in [29].

# Bibliography


[1] Beylkin, G.; Coifman, R. R.; Rokhlin, V.: *Fast wavelet transforms and numerical algorithms I*, Commun. Pure Appl. Math. **44**, (1991), pp. 141-183.

[2] Bittner, K.: *On the stability of compactly supported biorthogonal spline wavelets*, Approximation Theory **XII**, San Antonio, (2007), pp. 38-49.

[3] Bramble, J. H.; Cohen, A.; Dahmen, W.: *Multiscale Problems and Methods in Numerical Simulations*, Springer-Verlag Berlin Heidelberg, 2003.

[4] De Boor, C.: *A Practical Guide to Splines*, Springer Verlag, Berlin, 1978.

[5] Burrus, C.S.; Odegard, J. E.: *Coiflet Systems and Zero Moments*, IEEE Transactions on Signal Processing **46(3)**, (1998), pp. 761-766.

[6] Carnicer, J.M.; Dahmen, W.; Peña, J.M.: *Local decomposition of refinable spaces*, Appl. Comp. Harm. Anal. **6**, (1999), pp. 1-52.

[7] Černá, D.; Finěk, V.: *Construction of optimally conditioned cubic spline wavelets on the interval*, Adv. Comput. Math. **34(2)**, (2011), pp. 219-252.

[8] Černá, D.; Finěk, V.: *Cubic Spline Wavelets with Complementary Boundary Conditions*, Appl. Math. Comput. **219**, (2012), pp. 1853-1865.

[9] Černá, D.; Finěk, V.: *Approximate Multiplication in Adaptive Wavelet Methods*, Cent. Eur. J. Math. **11**, (2013), pp. 972-983.

[10] Černá, D.; Finěk, V.: *Quadratic Spline Wavelets with Short Support for Fourth-Order Problems*, Result. Math. **66**, (2014), pp. 525-540.

[11] Černá, D.; Finěk, V.: *Cubic Spline Wavelets with Short Support for Fourth-Order Problems*, Appl. Math. Comput. **243**, (2014), pp. 44-56.

[12] Černá, D.; Finěk, V.: *Wavelet basis of cubic splines on the hypercube satisfying homogeneous boundary conditions*, Int. J. Wavelets Multi. **13(3)**, (2015), pp. 1550014/1-21.

[13] Černá, D.; Finěk, V.: *On a sparse representation of a $n$-dimensional Laplacian in wavelet coordinates*, Result. Math., DOI 10.1007/s00025-015-0488-5, (2015).

[14] Černá, D.; Finěk, V.; Najzar, K.: *On the exact values of coefficients of Coiflets*, Cent. Eur. J. Math. **6(1)**, (2008), pp. 159-170.

[15] Christensen, O.: *An Introduction to Frames and Riesz Bases*, Applied and Numerical Harmonic Analysis, Birkhäuser, Boston, 2003.

[16] Chui, C.K.: *An Introduction to Wavelets*, Academic Press, Boston, 1992.

[17] Ciarlet, P.: *Finite Element Method for Elliptic Problems*, Society for Industrial and Applied Mathematics. Philadelphia, PA, USA (2002).

[18] Cohen, A.: *Numerical Analysis of Wavelet methods*, Studies in Mathematics and its Applications **32**, Elsevier, Amsterdam, 2003.

[19] Cohen, A.; Dahmen, W.; DeVore, R.: *Adaptive Wavelet Schemes for Elliptic Operator Equations - Convergence Rates*, Math. Comput. **70**, (2001), pp. 27-75.

[20] Cohen, A.; Dahmen, W.; DeVore, R.: *Adaptive Wavelet Methods II - Beyond the elliptic case*. Found. Math. **2**, (2002), pp. 203-245.

[21] Cohen, A.; Daubechies, I.; Feauveau, J.-C.: *Biorthogonal Bases of Compactly Supported Wavelets*, Comm. Pure and Appl. Math. **45**, (1992), pp. 485-560.

[22] Cohen, A.; Daubechies, I.; Vial, P.: *Wavelets on the Interval and Fast Wavelet Transforms*, Appl. Comp. Harm. Anal. **1**, (1993), pp. 54-81.

[23] Dahmen, W.: *Stability of Multiscale Transformations*, J. Fourier Anal. Appl. **4**, (1996), pp. 341-362.

[24] Dahmen, W.; Kunoth, A.; Urban, K.: *Biorthogonal Spline Wavelets on the Interval - Stability and Moment Conditions*, Appl. Comp. Harm. Anal. **6**, (1999), pp. 132-196.

[25] Dahmen, W.; Stevenson, R.: *Element-by-Element Construction of Wavelets Satisfying Stability and Moment Conditions*, SIAM J. Numer. Anal. **37**, (1999), pp. 319-352.

[26] Daubechies, I.: *Ten Lectures on Wavelets*, SIAM Publ., Philadelphia, 1992.

[27] Daubechies, I.: *Orthonormal bases of compactly supported wavelets II, Variations on a theme*, SIAM J. Math. Anal. **24(2)**, (1993), pp. 499-519.

[28] Dijkema, T. J.; Schwab, Ch.; Stevenson, R.: *An adaptive wavelet method for solving high-dimensional elliptic PDEs*, Constr. Approx. **30**, pp. 423-455 (2009).

[29] Dijkema, T. J.; Stevenson, R.: *A sparse Laplacian in tensor product wavelet coordinates*, Numer. Math. **115**, pp. 433-449 (2010).

[30] Eirola, T.: *Sobolev characterization of compactly supported wavelets*, SIAM J. Math. Anal. **23**, (1992), pp. 1015-1030.

[31] Griebel, M.; Oswald, P.: *Tensor product type subspace splittings and multilevel iterative methods for anisotropic problems*, Adv. Comput. Math. **4**, pp. 171-206 (1995).

[32] Lawton, W.M.: *Necessary and sufficient conditions for constructing orthonormal wavelet bases*, J. Math. Phys. **32(1)**, (1991), pp. 57-61.

[33] Lemarié, P.G.: *On the Existence of Compactly Supported Dual Wavelets*, Appl. Comp. Harm. Anal. **3**, (1997), pp. 117-118.

[34] Monzón, L.; Beylkin, G.; Hereman, W.: *Compactly supported wavelets based on almost interpolating and nearly linear phase filters (coiflets)*, Appl. Comp. Harm. Anal. **7**, (1999), pp. 184-210.

[35] Jia, R.Q.: *Bessel sequences in Sobolev space*, Appl. Comp. Harm. Anal. **20**, (2006), pp. 298-311.

[36] Jia, R.Q.; Zhao, W.: *Riesz bases of wavelets and applications to numerical solutions of elliptic equations*, Math. Comput. **80**, pp. 1525-1556 (2011).

[37] Primbs, M.: *New stable biorthogonal spline wavelets on the interval*. Result. Math. **57**, pp. 121-162 (2010).

[38] Ron A.; Shen, Z.: *Sobolev regularity of refinable functions*, Journal of Approximation Theory **106(2)**, (2000), pp. 185-225.

[39] Stevenson, R.: *Adaptive solution of operator equations using wavelet frames*, SIAM J. Numer. Anal. **41**, (2003), pp. 1074-1100.

[40] Stevenson, R.: *Locally supported, piecewise polynomial biorthogonal wavelets on nonuniform meshes*, Constr. Approx. **19(4)**, (2003), pp. 477-508.

[41] Villemoes, L.F.: *Energy moments in time and frequency for two-scale difference equation solutions and wavelets*, SIAM J. Math. Anal. **23**, (1992), pp.1519-1543.

[42] Walnut D.F.: *An Introduction to Wavelet Theory, Applied and Numerical Harmonic Analysis*, Birkhäuser, Boston, 2002.

[43] Wojtaszczyk, P.: *A Mathematical Introduction to Wavelets*, Cambridge University Press, 1997.

CESJ 1 (2003) 1–15 *Central European Science Journals*

# On the exact values of coefficients of coiflets

Dana Černá *[1,2], Václav Finěk †[1], Karel Najzar ‡[2]

[1] *Department of Mathematics and Didactics of Mathematics, Technical University of Liberec, Hálkova 6, Liberec 1, 461 17, Czech Republic*

[2] *Department of Numerical Mathematics, Charles University, Prague Ke Karlovu 3, Prague 2, 121 16, Czech Republic*

**Abstract:** In 1989, R. Coifman suggested the design of orthonormal wavelet systems with vanishing moments for both the scaling and the wavelet functions. They were first constructed by I. Daubechies [16, 15] and she named them coiflets. In this paper, we propose a system of necessary conditions which is redundant free and more simple than the known system due to elimination of some quadratic conditions, thus a construction of coiflets is simplified and enables us to find the exact values of the scaling coefficients of coiflets up to length 8 and two further with length 12. Furthermore for scaling coefficients of coiflets up to length 14 we obtain two quadratic equations, which can be transformed to polynomial of degree 4 and there is an algebraic formula to solve them.


*Keywords: orthonormal wavelet, coiflet, exact value of filter coefficients*
*MSC (2000): 65T60*

## 1 Introduction

Approximation properties of multiresolution analysis and the smoothness of wavelet and the scaling function depend on the number of vanishing wavelet moments. In [14] Daubechies constructed orthonormal wavelets with arbitrary number $N$ of vanishing wavelet moments and the minimal length of support $2N-1$. The filter coefficients were computed there by an analytical method and exact values could be found only for

* E-mail: dana.cerna@tul.cz
† E-mail: vaclav.finek@tul.cz
‡ E-mail: knaj@karlin.mff.cuni.cz

filters up to length 6. In [26] Shann and Yen calculated the exact values of filter coefficients of Daubechies wavelets of length 8 and 10. Other approaches for constructing Daubechies wavelets which enables to find exact values of some coefficients can be found in [9, 10, 23, 24].

In addition to the orthogonality, compact support and vanishing wavelet moments, Coifman has suggested that also requiring vanishing scaling moments has some advantages. In practical applications these wavelets are useful due to their 'nearly linear phase' and 'almost interpolating property', see [22]. Daubechies created coiflets by setting an equal number $N$ of vanishing wavelet moments and vanishing scaling moments for even $N$ and the length of support $3N$, see [16, 15]. It was noticed in [4] that these coiflets has one additional vanishing scaling moment than is imposed. Tian constructed coiflets with $N$ vanishing moments for odd $N$ and the length of support $3N-1$ in [27, 29]. Burrus and Odegard constructed coiflets with $N$ vanishing moments for odd $N$ and the length of support $3N+1$ which has two additional vanishing scaling moments, see [7]. In this paper the computation of exact values of filter coefficients of coiflets up to filter length 14 is presented.

There exist a number of coiflet filter design methods, such as Newton's method [16, 25] or iterative numerical optimization [7]. These methods enable to derive one particular solution for each system and the convergence and the obtained solution depends on the initial starting point, thus it is difficult to find all possible solutions. Moreover, the coefficients for length greater than 16 are given with less precision due to the roundoff error [15]. As an alternative one can use Gröbner basis method [1, 6, 21]. This method is geared toward solving a polynomial system of equations with finite solutions. The idea consists of finding a new set of equations equivalent to the original set, which can be solved more easily. The advantage of such an approach is that solutions can be computed to arbitrary precision and that in some cases it gives all possible solutions for a given system of polynomial equations. In this paper we derive a redundant free and simplified system of equations and then aplly Gröbner basis method. By this approach we are able to find some exact values of filter coefficients and to find all possible solutions for filters up to length 20.

## 2 Preliminaries

The scaling function $\phi$, which generates a coiflet, is constructed as the solution of scaling equation

$$\phi = 2 \sum_{k\in\mathbb{Z}} h_k \phi(2\cdot -k), \tag{1}$$

where scaling coefficients $\{h_k\}$ are determined so that the corresponding scaling functions and wavelets have required properties.

**Definition 2.1.** An orthonormal wavelet $\psi$ with compact support is called a coiflet of order $N$, if the following conditions are satisfied:

$$i)\ \int_{-\infty}^{\infty} x^n \psi(x)\, dx = 0 \quad \text{for} \quad n = 0, \dots N-1,$$

$$ii)\ \int_{-\infty}^{\infty} x^n \phi(x)\, dx = \delta_n \quad \text{for} \quad n = 0, \dots N-1,$$

where $\phi$ is scaling function corresponding to $\psi$ and $\delta_n$ is Kronecker delta, i.e. $\delta_0 = 1$ and $\delta_n = 0$ for $n \neq 0$.

Since also a length of support plays a role, it is common to consider a wavelet satisfying $i)$ and $ii)$ which has the minimal length of support. The existence of coiflet for an arbitrary order $N$ is still an open question. We rewrite this definition in terms of filter coefficients $\{h_k\}$. It is known that for orthonormal wavelet with compact support a number of filter coefficients is even number, we denote it by $2M$.

**Lemma 2.2.** Let $\{h_k\}_{k=N_1}^{N_2}$ be the real coefficients, $N_2 = N_1 + 2M - 1$. If the orthonormal wavelet corresponding to the scaling function $\phi(\cdot) = 2\sum_{k=N_1}^{N_2} h_k\, \phi(2\cdot - k)$ is a coiflet of order $N$, then the following three conditions are satisfied:

$$i)\ \delta_m = 2 \sum_{j=0}^{N_2 - N_1 - 2m} h_{N_1+j} h_{N_1+2m+j} \quad \text{for} \quad 0 \le m \le M-1,$$

$$ii)\ \sum_{k=N_1}^{N_2} h_k k^n = \delta_n \quad \text{for} \quad 0 \le n \le N-1,$$

$$iii)\ \sum_{k=N_1}^{N_2} (-1)^k h_k k^n = 0 \quad \text{for} \quad 0 \le n \le N-1.$$

Condition $i)$ is necessary but not sufficient for wavelet to be orthonormal. Conditions $ii)$ and $iii)$ are equivalent to vanishing wavelet and vanishing scaling function moments, respectively. In summary, conditions in Lemma 2.2 are only necessary. It is known that they are not sufficient to generate a coiflet system. There exist functions given by (1) with filter coefficients satisfying $i) - iii)$ from Lemma 2.2 which are very rough. Hence after finding coefficients satisfying $i) - iii)$ orthonormality should be verified, for example using Cohen [11] or Lawton [20] condition. There are typically more than one wavelet satisfying these conditions and some of them, despite zero wavelet moments, are very rough. Likewise, in spite of zero scaling function moments, some are not at all symmetric. In practical applications the most regular wavelet or the wavelet with the most symetrical scaling function is typically chosen.

## 3 Construction

It is well known that coiflets have more vanishing scaling moments than required in above definition. This was first noted by G. Beylkin at al. in [4]. In this paper, we derive redundant free and simpler definition of coiflets. The key component of our approach is formed by the following Theorem:

**Theorem 3.1.** Let $N_2 = N_1 + 2M - 1$ then

$$\delta_m = 2 \sum_{j=N_1}^{N_2-2m} h_j h_{j+2m} \quad \text{for} \quad 0 \le m \le M-1 \tag{2}$$

is equivalent to

$$\frac{1}{2}\,\delta_n = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i}) \quad \text{for} \quad 0 \le n \le M-1, \tag{3}$$

where

$$a_i = \sum_{k=0}^{M-1} (N_1 + 2k)^i h_{N_1+2k} \qquad \text{and} \qquad b_i = \sum_{k=0}^{M-1} (N_1 + 2k + 1)^i h_{N_1+2k+1}. \tag{4}$$

**Proof.** Since the condition (2) is equivalent to condition

$$|m(\omega)|^2 + |m(\omega + \pi)|^2 = 1, \tag{5}$$

where

$$m(\omega) = \sum_{k=N_1}^{N_2} h_k e^{-ik\omega},$$

we can repeat the proof of Theorem 3.1 in [19] with some obvious changes. □

Due to Theorem 3.1 we are now able to impose necessary conditions on filter coefficients to generate a coiflet which are equivalent to conditions from Lemma 2.2 and the arising system is without redundant conditions.

**Corollary 3.2.** Let $\{h_k\}_{k=N_1}^{N_2}$ be the real coefficients, $N_2 = N_1 + 2M - 1$ and let $a_i$ and $b_i$ be defined by (4). Then conditions $i) - iii)$ from Lemma 2.2 are equivalent to the following conditions:

$i*)$ $0 = \sum_{i=0}^{2n} \binom{2n}{i} (-1)^i (a_i a_{2n-i} + b_i b_{2n-i}) \quad \text{for} \quad N \le n \le M-1,$

$ii*)$ $a_0 = b_0 = \frac{1}{2},$

$iii*)$ $a_n = b_n = 0 \quad \text{for} \quad 1 \le n \le N-1,$

$iv*)$ $a_{2n} + b_{2n} = 0 \quad \text{for} \quad N \le 2n \le 2N-2.$

**Proof.** It is clear that *ii*) and *iii*) are equivalent to *ii*∗) and *iii*∗). The rest follows from Theorem 3.1. □

The consequence of this Corollary is that the minimal length of support of coiflet of order $N$ is $3N$ for even $N$ and $3N-1$ for odd $N$ and that some coiflets have more vanishing moments than is imposed. Thus, we have three classes of coiflets, see Table 1.

**Table 1** The length of filter $2M$, the number of vanishing scaling and wavelet moments for coiflet of order $N$

| N | 2M | number of vanishing scaling moments | | number of vanishing wavelet moments | |
|---|---|---|---|---|---|
| | | set | actual | set | actual |
| even | 3N | N | N+1 | N | N |
| odd | 3N-1 | N | N | N | N |
| odd | 3N+1 | N+1 | N+2 | N | N |

Now we further simplify the system by replacing some quadratic conditions by linear ones.

**Lemma 3.3.** Let $a_i$, $b_i$ be defined by (4). Then $a_i$ for $i \geq M$ is linear combination of $a_0, \ldots a_{M-1}$ and $b_i$ for $i \geq M$ is linear combination of $b_0, \ldots b_{M-1}$.

**Proof.** Coefficients $h_{N_1}, h_{N_1+2}, \ldots h_{N_1+2M-2}$ are solution of the system of linear algebraic equations (4). Since the matrix of this system is regular, the solution exists and is unique. $a_i$ is a linear combination of $h_{N_1}, h_{N_1+2}, \ldots h_{N_1+2M-2}$ and thus $a_i$ for $i \geq M$ is a linear combination of $a_i$ for $0 \leq i \leq M-1$:

$$a_i = c_{i0}a_0 + c_{i1}a_1 + \ldots c_{iM-1}a_{M-1},$$

where the coefficients of this linear combinations are given by

$$\begin{pmatrix} 1 & N_1 & N_1^2 & \ldots & N_1^{M-1} \\ 1 & N_1+2 & (N_1+2)^2 & \ldots & (N_1+2)^{M-1} \\ \vdots & & & & \vdots \\ 1 & N_1+2M-2 & (N_1+2M-2)^2 & \ldots & (N_1+2M-2)^{M-1} \end{pmatrix} \begin{pmatrix} c_{i0} \\ c_{i1} \\ \vdots \\ c_{iM-1} \end{pmatrix} = \begin{pmatrix} N_1^i \\ (N_1+2)^i \\ \vdots \\ (N_1+2M-2)^i \end{pmatrix}.$$

The situation for $b_i$ is similar. □

Now we summarize the procedure of construction which enables us to find exact values of coefficients of coiflets up to length of support 14:

**1.** For given $N$ take the system of algebraic equations given by Corollary 3.2.

**2.** Replace $a_M, \dots a_{2M-2}$ by linear combinations of $a_0, \dots a_{M-1}$ and $b_M, \dots b_{2M-2}$ by linear combinations of $b_0, \dots b_{M-1}$.

**3.** Solve the arising system for $a_0, \dots a_{M-1}, b_0, \dots b_{M-1}$. For greater $N$ use Gröbner basis method to simplify the system.

**4.** Compute filter coefficients $h_{N_1}, \dots, h_{N_2}$ by solving the system of linear algebraic equations (4).

## 4 Examples

At last we provide two examples to illustrate our approach based on Corollary 3.2.

**Example 4.1.** For $N = 4$ and $N_1 = -5$, the following system will be obtained

$$a_0 = b_0 = \frac{1}{2} \quad \text{and} \quad a_1 = a_2 = a_3 = b_1 = b_2 = b_3 = 0,$$

$$a_4 + b_4 = 0 \quad \text{and} \quad a_6 + b_6 = 0, \tag{6}$$

$$a_8 + b_8 + 140\, b_4^2 = 0 \quad \text{and} \quad a_{10} + b_{10} + 840\, b_4\, b_6 - 252(a_5^2 + b_5^2) = 0. \tag{7}$$

Now $a_6$, $a_8$, $a_{10}$, $b_6$, $b_8$, $b_{10}$ are linear combinations of $a_0 \dots a_5$, $b_0 \dots b_5$. We find these linear combinations and substitute them to (6) and (7). Then after simplification we obtain system

$$-135 + 12b_4 + 8b_4^2 = 0, \quad a_4 + b_4 = 0,$$

$$75 - 10b_4 + 4b_5 = 0, \quad 32a_5^2 + 12300b_4 - 28575 = 0.$$

In this case we can easily find both real solutions in closed form. See Table 2 and Table 3.

**Example 4.2.** For $N = 5$ and $N_1 = -5$, the following system will be obtained

$$a_0 = b_0 = \frac{1}{2} \quad \text{and} \quad a_1 = a_2 = a_3 = a_4 = b_1 = b_2 = b_3 = b_4 = 0,$$

$$a_6 + b_6 = 0 \quad \text{and} \quad a_8 + b_8 = 0, \tag{8}$$

$$a_{10} + b_{10} - 252(a_5^2 + b_5^2) = 0 \quad \text{and} \quad a_{12} + b_{12} - 1584\, b_5\, b_7 - 1584\, a_5\, a_7 + 924(a_5^2 + b_5^2) = 0. \tag{9}$$

Now $a_7$, $a_8$, $a_{10}$, $a_{12}$, $b_6$, $b_8$, $b_{10}$, $b_{12}$ are linear combinations of $a_0 \dots a_6$, $b_0 \dots b_6$. We find these linear combinations and substitute them to (8) and (9). Consequently we simplify arising system and finally we compute its Gröbner bases. The following system is obtained:

$$11419648\, b_5^4 + 246374400\, b_5^3 - 13765248000\, b_5^2 - 497539800000\, b_5 - 4303042734375 = 0,$$

$$298890000\, a_5 - 5709824\, b_5^3 + 3945600\, b_5^2 + 6931764000\, b_5 + 94943559375 = 0,$$

$$8\,a_6 + 64\,b_5 + 525 = 0, \qquad -525 - 64\,b_5 + 8\,b_6 = 0.$$

Then by using an algebraic formula for the solution of polynomials of degree 4 we obtain two different real roots:

$$b_5 = \frac{15\left(\sqrt{15}u^{3/4} - 4010u^{1/6}v^{1/4} \pm \sqrt{15}\sqrt{w}\right)}{11152u^{1/6}v^{1/4}},$$

where

$$u = 4854802096 + 369\sqrt{15}\sqrt{66685436848043}, \quad v = 8475076\,u^{1/3} + 697\,u^{2/3} - 3366028373,$$

$$w = 16950152\,u^{1/3}\sqrt{v} - 697\sqrt{v}\,u^{2/3} + 3366028373\sqrt{v} + 13383342756\sqrt{15}\sqrt{u}.$$

Once we have the values of $b_5$, we simply find $a_5$, $a_6$, and $b_6$. And finally we transform coefficients $a_i$ and $b_i$ to scaling coefficients $h_i$.

## 5 Properties of coiflets

Let us now mention the properties of constructed wavelets. It is well-known that the approximation properties depend on the number of vanishing wavelet moments. More precisely, let $P_j f$ be an approximation of $f \in L^2(\mathbb{R})$ on level $j$, i.e.

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{j,k}\rangle\, \phi_{j,k} \tag{10}$$

and for $J < j$, it holds

$$P_j f = \sum_{k\in\mathbb{Z}} \langle f, \phi_{J,k}\rangle\, \phi_{J,k} + \sum_{l=J}^{j-1}\sum_{k\in\mathbb{Z}} \langle f, \psi_{l,k}\rangle\, \psi_{l,k}, \tag{11}$$

where $\phi_{l,k} = 2^{l/2}\phi\left(2^l\cdot - k\right)$, $\psi_{l,k} = 2^{l/2}\psi\left(2^l\cdot - k\right)$ for $l, k \in \mathbb{Z}$. Let us further denote $I_{l,k} = \text{supp}\ \phi_{l,k}$, $J_{l,k} = \text{supp}\ \psi_{l,k}$. Wavelet coefficients satisfy

$$\langle f, \psi_{l,k}\rangle = \int_{-\infty}^{\infty} f(x)\, 2^{l/2}\psi\left(2^l x - k\right) dx. \tag{12}$$

and if $f \in C^N(J_{l,k})$, then expanding $f$ about $\frac{k}{2^l}$ by Taylor's formula, it follows that for all $x \in J_{l,k}$,

$$f(x) = f\left(\frac{k}{2^l}\right) + f'\left(\frac{k}{2^l}\right)\left(x - \frac{k}{2^l}\right) + \ldots + \frac{f^{(N-1)}\left(\frac{k}{2^l}\right)}{(N-1)!}\left(x - \frac{k}{2^l}\right)^{N-1} + \frac{f^{(N)}(\xi)}{N!}\left(x - \frac{k}{2^l}\right)^N, \tag{13}$$

where $\xi$ depends on $x$ and belongs to the interval $J_{l,k}$. If $\psi$ has $N$ vanishing moments, i.e. condition $i)$ in Definition 2.1 is satisfied, then the first $N$ terms don't contribute and

$$|\langle f, \psi_{l,k}\rangle| = \left|\int_{-\infty}^{\infty} \frac{f^{(N)}(\xi(x))}{N!}\left(x - \frac{k}{2^l}\right)^N 2^{l/2}\psi\left(2^l x - k\right) dx\right| \le C2^{-l(N+1/2)}, \tag{14}$$

where

$$C = \frac{\max_{\xi \in J_{l,k}} \left| f^{(N)}(\xi) \right|}{N!} \int_{J_{l,k}} |y|^N \psi(y)\, dy. \tag{15}$$

Thus, for $l$ large, the wavelet coefficients are small except these which are near singularities of the function $f$ or its derivatives. Small coefficients can be set to zero and the function $f$ can be represented by a small number of coefficients. This compression property of wavelets has many applications. Most important are data compression, signal analysis and efficient adaptive schemes for PDE's. Note that more vanishing wavelet moments implies a faster decay of wavelet coefficients and that only local smoothness of the function $f$ is involved in the above estimate. It was observed in [2] that also regularity of the scaling function plays a role. We confirmed in our experiments that this is true for coiflets as well. As an example, let us consider

$$\begin{aligned} f(x) &= x^5 && \text{if} \quad 0 \le x \le 0.5, \\ &= (1-x)^5 && \text{if} \quad 0.5 < x \le 1, \\ &= 0 && \text{otherwise}, \end{aligned}$$

and its $n$-term approximation

$$f_n(x) = \sum_{\lambda=(l,k)\in\Lambda_\phi^n} \langle f, \phi_\lambda \rangle \phi_\lambda + \sum_{\lambda=(l,k)\in\Lambda_\psi^n} \langle f, \psi_\lambda \rangle \psi_\lambda, \tag{16}$$

where $\Lambda_\phi^n \subset \{\lambda = (J,k), k \in \mathbb{Z}\}$, $\Lambda_\psi^n \subset \{\lambda = (l,k), J \le l < j, k \in \mathbb{Z}\}$ and $\Lambda_\phi^n \cup \Lambda_\psi^n$ is the set of indexes of the $n$ largest coefficients. In our case, the coarsest level is $J = 3$, the finest level is $j = 9$ and the number of preserved coefficients is $n = 50$. The function $f$ has sharp derivative near the point $x = 0.5$ and the approximation is automatically refined near this point. Errors of approximation for some of the constructed coiflets are shown in Table 2. We can see that the most regular coiflet of prescribed order gives the best result.

The significance of vanishing scaling moments highly depends on the type of application. In [16], it is proved that all real orthonormal wavelets with compact support are asymmetric. However, vanishing scaling moments result in 'almost symmetry' of the scaling function and filter. In image coding, more symmetry would result in greater compressibility for the same perceptual error and it makes easier to deal with the boundaries of the image. Vanishing scaling moments also causes 'nearly linear phase', which is a desired quality in many applications, e.g. transmission of audio and video signals, because it doesn't cause phase distortion. In numerical analysis, vanishing scaling moments are important due to their 'almost interpolating property'. It means that any $f \in C_0^N(\mathbb{R})$ can be approximated by

$$f_j = 2^{-j/2} \sum_{k\in\mathbb{Z}} f\left(\frac{k}{2^j}\right) \phi_{j,k} \tag{17}$$

and if the number of vanishing scaling and wavelet moments is $N$ then this approximation satisfies the following estimate

$$\|f - f_j\| \le C 2^{-jN}, \tag{18}$$

where $C$ depends only on $f$ and the scaling function $\phi$, see [28]. Due to this property, some types of operators can be treated efficiently. Thus coiflets have some interesting properties and for some applications are more suitable than orthonormal wavelets with vanishing wavelet moments only. The price to pay is of course the length of support, which can make the computation more expensive. We should also mention that we can obtain symmetric wavelets by giving up orthonormality. Symmetric biorthogonal wavelets were constructed in [12], and construction of biorthogonal coiflets can be found in [28, 29]. However, there are applications where orthogonality plays a role and the disadvantage of biorthogonal wavelets is their bad stability when adapted to the interval, see [5, 13].

In literature, one can find coiflets which are the most symmetrical among all coiflets of given order and given length of support, see [7, 15, 16, 27, 29]. As we could see above, these coiflets needn't be the best and other solutions of equations given in Corollarry 3.2 may be better suited for some type of applications. Typically the most regular coiflet for given order $N$ has the best compression property and due to almost interpolating property and ability to generate a stable wavelet basis on bounded domain it seems to be very well suited for some applications, e.g. numerical solution of PDE's.

**Table 2** Error of approximation of the function $f$ by 50 coefficients for coiflets of order $N$, length of support $2M$ and Sobolev exponent of smoothness $\alpha$

| N | 2M | $\alpha$ | $L^\infty$ of error $\times 10^{-6}$ | $L^2$ norm of error $\times 10^{-7}$ | $H^1$ seminorm of error $\times 10^{-4}$ |
|---|---|---|---|---|---|
| 1 | 4 | 0.604 | 743 | 1986 | 1358 |
| 1 | 4 | 0.050 | 2800 | 7332 | 5642 |
| 2 | 6 | 0.041 | 402 | 978 | 706 |
| 2 | 6 | 1.232 | 44 | 116 | 46 |
| 2 | 6 | 0.590 | 184 | 469 | 234 |
| 2 | 6 | 1.022 | 83 | 200 | 87 |
| 3 | 8 | 0.147 | 103 | 225 | 137 |
| 3 | 8 | 1.775 | 2 | 6 | 1 |
| 3 | 8 | 1.422 | 20 | 31 | 13 |
| 3 | 8 | 0.936 | 44 | 97 | 33 |
| 3 | 8 | 1.464 | 15 | 33 | 10 |
| 3 | 8 | 1.773 | 3 | 5 | 1 |

## 6 Conclusion

The arising system from the Corollary 3.2 is redundant-free, more simple (due to elimination of some quadratic conditions) and enables to find directly the exact values of the scaling coefficients of coiflets up to length 8 and two further with length 12 in closed form. The results are given in Table 3, Table 4 and Table 5. We verified orthonormality by Lawton criterion, all the results correspond to orthonormal scaling function. As mentioned earlier, the solutions are not of the same quality, because also smoothness and symmetry plays a role. For this reason the most symmetrical scaling function among all scaling functions of order $N$ is denoted in Tables and the Sobolev exponents of smoothness are computed by method from [17, 31]. Furthermore for remaining coiflets up to length 14 we obtain two quadratic equations of two variables, which can be transformed to polynomial of degree 4 and there is an algebraic formula to find solutions in closed form. These solutions we do not provide because of their length and complicated structure. Moreover, one can use our approach to find all possible solutions to given system up to the length of filter 20. For longer filters the computation failed, because the coefficients of polynomials in Gröbner basis were too large numbers.

**Acknowledgements.** This research is partially supported by grant No IGS 116/5130/1 financed by TU Liberec. The first author was supported by project LC06024 and by grant No MSM 0021620839 financed by Ministry of Education, Youth and Sports of the Czech Republic and the second author was supported by project 1M06047 of Ministry of Education, Youth and Sports of the Czech Republic.

## References

[1] Adams, W.W.; Loustaunau, P.: *An Introduction to Gröbner Bases.* American Mathematical Society, 1994.

[2] Antonini, M.; Barlaud, M.; Mathieu, P.; Daubechies, I.: "Image coding using wavelet transforms", *IEEE Trans. ImageProcess.*, Vol. 1, (1992), pp. 205-220.

[3] Beylkin, G.: "On the representation of operators in bases of compactly supported wavelets", *SIAM J. Numer. Anal.*, Vol. 29, no. 6, (1992), pp. 1716-1740.

[4] Beylkin, G.; Coifman, R. R.; Rokhlin, V.: "Fast wavelet transforms and numerical algorithms I", *Commun. Pure Appl. Math.*, Vol. 44, (1991), pp. 141-183.

[5] Bittner, K.; Urban, K.: "Adaptive wavelet methods using semiorthogonal spline wavelets: Sparse evaluation of nonlinear functions." Univ. Ulm. Preprint 2005. To appear in Appl. Comp. Harm. Anal.

[6] Buchberger, B.: *Ein Algorithmus zum Auffinden der Basiselemente des Restklassenringes nach einem nulldimensionalen Polynomideal (An algorithm for finding a basis for the residue class ring of a zero-dimensional polynomial ideal).* PhD thesis, Univ. of Inssbruck, Austria, 1965.

[7] Burrus, C.S.; Odegard, J. E.: "Coiflet Systems and Zero Moments", *IEEE Transactions on Signal Processing*, Vol. 46, no. 3., (1998), pp. 761-766.

[8] Burrus, C.S.; Gopinath, R. A.: "On the moments of the scaling function $\psi_0$", In: *Proc. IEEE Int. Symp. Circuits Syst.* , San Diego, CA, (1992), pp. 963-966.

[9] Černá, D.; Finěk, V.: "On the computation of scaling coefficients of Daubechies wavelets." *Cent. European J. Math.*, Vol. 2, no. 4, (2004), pp. 399-419.

[10] Chyzak, F.; Paule, P.; Scherzer, O.; Schoisswohl, A.; Zimmermann, B.: "The construction of orthonormal wavelets using symbolic methods and a matrix analytical approach for wavelets on the interval." *Experiment. Math.*, Vol. 10, (2001), pp. 67-86.

[11] Cohen, A.: "Ondelettes, analyses multirésolutions et filtres miroir en quadrature", Ann. Inst. H. Poincaré, Anal. non linéaire, Vol. 7, (1990), pp. 439-459.

[12] Cohen, A.; Daubechies, I.; Feauveau, J.C.: "Biorthogonal bases of compactly supported wavelets." *Comm. Pure Appl. Math.*, Vol.45, (1992), pp. 485-500.

[13] Dahmen, W.; Kunoth, A.; Urban, K.: "Biorthogonal spline wavelets on the interval - stability and moment conditions." *Appl. Comp. Harm. Anal.*, Vol. 6; (1999), pp. 132-196.

[14] Daubechies, I.: "Orthonormal bases of compactly supported wavelets", *Commun. Pure Appl. Math.*, Vol. 41, (1988), pp. 909-996.

[15] Daubechies, I.: "Orthonormal bases of compactly supported wavelets II, Variations on a theme", *SIAM J. Math. Anal.*, Vol. 24, no. 2, (1993), pp. 499-519.

[16] Daubechies, I.: "Ten Lectures on Wavelets", *SIAM Publ.*, Philadelphia, 1992.

[17] Eirola, T.: "Sobolev characterization of compactly supported wavelets", *SIAM J. Math. Anal.*, Vol. 23, (1992), pp.1015-1030.

[18] Finěk, V.: "Approximation properties of wavelets and relations among scaling moments", *Numerical Functional Analysis and Optimization*, Vol. 25, no. 5-6, (2004), pp. 503-513.

[19] Finěk, V.: "Approximation properties of wavelets and relations among scaling moments II", *Cent. European J. Math.*, Vol. 2, no. 4, (2004), pp. 605-613.

[20] Lawton, W.M.: "Necessary and sufficient conditions for constructing orthonormal wavelet bases", *J. Math. Phys.*, Vol. 32, no. 1, (1991), pp. 57-61.

[21] Lebrun, J.; Selesnick, I.: "Grobner bases and wavelet design." *J. Symb. Comp.*, Vol. 37, no. 2, (2004), pp. 227-259.

[22] Monzón, L., Beylkin, G., Hereman, W.: "Compactly supported wavelets based on almost interpolating and nearly linear phase filters (coiflets)", *Applied and Computational Harmonic Analysis 7*, (1999), pp. 184-210.

[23] Regensburger, G.; Scherzer, O.: "Symbolic computation for moments and filter coefficients of scaling functions." *Ann. Comb.*, Vol. 9, no. 2, (2005), pp. 223-243.

[24] Regensburger, G.: "Parametrizing compactly supported orthonormal wavelets by discrete moments." *Appl. Alg. Eng. Comm. Comp.*, no. 4., (2007), pp.

[25] Resnikoff, H.L.; Wells, R.O.: *Wavelet analysis. The scalable structure of information.* Springer

[26] Shann, W.C.; Yen, C.C.: "On the exact values of orthonormal scaling coefficients of length 8 and 10", *Applied and Computational Harmonic Analysis*, Vol. ,(1998), pp. .

[27] Tian, J.: *The mathematical theory and applications of biorthogonal Coifman wavelet systems*, Ph.D. dissertation, Rice Univ., Houston, TX, 1996.

[28] Tian, J.; Wells, R.O. Jr.: "Vanishing moments and biorthogonal Coifman wavelet systems." *Proc. of 4th International Conference on Mathematics in Signal Processing*, University of Warwick, England, (1997).

[29] Tian, J.; Wells, R.O. Jr.: "Vanishing moments and wavelet approximation", *Tech. Rep. CML TR-9501*, Comput. Math. Lab., Rice Univ., Houston, TX, 1995.

[30] Unser, M.: "Approximation power of biorthogonal wavelet expansions", *IEEE Trans. Signal Processing*, Vol. 44, no. 3., (1996), pp. 519-527.

[31] Villemoes, L.F.: "Energy moments in time and frequency for two-scale difference equation solutions and wavelets", *SIAM J. Math. Anal.*, Vol. 23, (1992), pp.1519-1543.

**Table 3** Scaling coefficients of coiflets of order $N$, length of filter $2M$ and Sobolev exponent $\alpha$

| | $n$ | $h_n$ | | $n$ | $h_n$ |
|---|---|---|---|---|---|
| $N=1, 2M=2$ | 0 | $\frac{1}{2}$ | | 1 | $\frac{7+\sqrt{7}}{16}$ |
| Haar wavelet | 1 | $\frac{1}{2}$ | | 2 | $\frac{1+\sqrt{7}}{16}$ |
| $N=1, 2M=4$ | -1 | $\frac{3}{8}-\frac{\sqrt{3}}{8}$ | | 3 | $\frac{-3-\sqrt{7}}{32}$ |
| $\alpha=0.604$ | 0 | $\frac{3}{8}+\frac{\sqrt{3}}{8}$ | $N=2, 2M=6$ | -2 | $\frac{1-\sqrt{7}}{32}$ |
| | 1 | $\frac{1}{8}+\frac{\sqrt{3}}{8}$ | $\alpha=1.022$ | -1 | $\frac{5+\sqrt{7}}{32}$ |
| | 2 | $\frac{1}{8}-\frac{\sqrt{3}}{8}$ | most symmetrical | 0 | $\frac{7+\sqrt{7}}{16}$ |
| $N=1, 2M=4$ | -1 | $\frac{3}{8}+\frac{\sqrt{3}}{8}$ | | 1 | $\frac{7-\sqrt{7}}{16}$ |
| $\alpha=0.050$ | 0 | $\frac{3}{8}-\frac{\sqrt{3}}{8}$ | | 2 | $\frac{1-\sqrt{7}}{16}$ |
| | 1 | $\frac{1}{8}-\frac{\sqrt{3}}{8}$ | | 3 | $\frac{-3+\sqrt{7}}{32}$ |
| | 2 | $\frac{1}{8}+\frac{\sqrt{3}}{8}$ | $N=3, 2M=8$ | -1 | $\frac{15}{64}+\frac{3\sqrt{1495}}{1664}$ |
| $N=2, 2M=6$ | -1 | $\frac{9+\sqrt{15}}{32}$ | $\alpha=0.147$ | 0 | $\frac{59}{128}-\frac{\sqrt{1495}}{832}$ |
| $\alpha=0.041$ | 0 | $\frac{13-\sqrt{15}}{32}$ | | 1 | $\frac{15}{64}-\frac{9\sqrt{1495}}{1664}$ |
| | 1 | $\frac{3-\sqrt{15}}{16}$ | | 2 | $\frac{15}{128}+\frac{3\sqrt{1495}}{832}$ |
| | 2 | $\frac{3+\sqrt{15}}{16}$ | | 3 | $\frac{5}{64}+\frac{9\sqrt{1495}}{1664}$ |
| | 3 | $\frac{1+\sqrt{15}}{32}$ | | 4 | $-\frac{15}{128}-\frac{3\sqrt{1495}}{832}$ |
| | 4 | $\frac{-3-\sqrt{15}}{32}$ | | 5 | $-\frac{3}{64}-\frac{3\sqrt{1495}}{1664}$ |
| $N=2, 2M=6$ | -1 | $\frac{9-\sqrt{15}}{32}$ | | 6 | $\frac{5}{128}+\frac{\sqrt{1495}}{832}$ |
| $\alpha=1.232$ | 0 | $\frac{13+\sqrt{15}}{32}$ | $N=3, 2M=8$ | -1 | $\frac{15}{64}-\frac{3\sqrt{1495}}{1664}$ |
| | 1 | $\frac{3+\sqrt{15}}{16}$ | $\alpha=1.775$ | 0 | $\frac{59}{128}+\frac{\sqrt{1495}}{832}$ |
| | 2 | $\frac{3-\sqrt{15}}{16}$ | | 1 | $\frac{15}{64}+\frac{9\sqrt{1495}}{1664}$ |
| | 3 | $\frac{1-\sqrt{15}}{32}$ | | 2 | $\frac{15}{128}-\frac{3\sqrt{1495}}{832}$ |
| | 4 | $\frac{-3+\sqrt{15}}{32}$ | | 3 | $\frac{5}{64}-\frac{9\sqrt{1495}}{1664}$ |
| $N=2, 2M=6$ | -2 | $\frac{1+\sqrt{7}}{32}$ | | 4 | $-\frac{15}{128}+\frac{3\sqrt{1495}}{832}$ |
| $\alpha=0.590$ | -1 | $\frac{5-\sqrt{7}}{32}$ | | 5 | $-\frac{3}{64}+\frac{3\sqrt{1495}}{1664}$ |
| | 0 | $\frac{7-\sqrt{7}}{16}$ | | 6 | $\frac{5}{128}-\frac{\sqrt{1495}}{832}$ |

**Table 4** Scaling coefficients of coiflets of order $N$, length of filter $2M$ and Sobolev exponent $\alpha$

| | $n$ | $h_n$ | | $n$ | $h_n$ |
|---|---|---|---|---|---|
| $N=3, 2M=8$ | -2 | $\frac{21}{640}-\frac{3\sqrt{31}}{320}$ | $N=3, 2M=8$ | -3 | $-\frac{1}{32}-\frac{\sqrt{7}}{128}$ |
| $\alpha=1.422$ | -1 | $\frac{51}{320}+\frac{3\sqrt{31}}{640}$ | $\alpha=1.773$ | -2 | $-\frac{3}{128}$ |
| | 0 | $\frac{257}{640}+\frac{9\sqrt{31}}{320}$ | most symmetrical | -1 | $\frac{9}{32}+\frac{3\sqrt{7}}{128}$ |
| | 1 | $\frac{147}{320}-\frac{9\sqrt{31}}{640}$ | | 0 | $\frac{73}{128}$ |
| | 2 | $\frac{63}{640}-\frac{9\sqrt{31}}{320}$ | | 1 | $\frac{9}{32}-\frac{3\sqrt{7}}{128}$ |
| | 3 | $\frac{-47}{320}+\frac{9\sqrt{31}}{640}$ | | 2 | $-\frac{9}{128}$ |
| | 4 | $\frac{-21}{640}+\frac{3\sqrt{31}}{320}$ | | 3 | $-\frac{1}{32}+\frac{\sqrt{7}}{128}$ |
| | 5 | $\frac{9}{320}-\frac{3\sqrt{31}}{640}$ | | 4 | $\frac{3}{128}$ |
| $N=3, 2M=8$ | -2 | $\frac{21}{640}+\frac{3\sqrt{31}}{320}$ | $N=4, 2M=12$ | -5 | $\frac{7}{1024}+\frac{\sqrt{31}}{1024}-\frac{\sqrt{336+82\sqrt{31}}}{2048}$ |
| $\alpha=0.936$ | -1 | $\frac{51}{320}-\frac{3\sqrt{31}}{640}$ | $\alpha=1.707$ | -4 | $\frac{7}{2048}-\frac{3\sqrt{31}}{2048}$ |
| | 0 | $\frac{257}{640}-\frac{9\sqrt{31}}{320}$ | | -3 | $-\frac{53}{1024}-\frac{3\sqrt{31}}{1024}+\frac{5\sqrt{336+82\sqrt{31}}}{2048}$ |
| | 1 | $\frac{147}{320}+\frac{9\sqrt{31}}{640}$ | | -2 | $-\frac{39}{2048}+\frac{11\sqrt{31}}{2048}$ |
| | 2 | $\frac{63}{640}+\frac{9\sqrt{31}}{320}$ | | -1 | $\frac{151}{512}+\frac{\sqrt{31}}{512}-\frac{5\sqrt{336+82\sqrt{31}}}{1024}$ |
| | 3 | $\frac{-47}{320}-\frac{9\sqrt{31}}{640}$ | | 0 | $\frac{555}{1024}-\frac{7\sqrt{31}}{1024}$ |
| | 4 | $\frac{-21}{640}-\frac{3\sqrt{31}}{320}$ | | 1 | $\frac{151}{512}+\frac{\sqrt{31}}{512}+\frac{5\sqrt{336+82\sqrt{31}}}{1024}$ |
| | 5 | $\frac{9}{320}+\frac{3\sqrt{31}}{640}$ | | 2 | $-\frac{47}{1024}+\frac{3\sqrt{31}}{1024}$ |
| $N=3, 2M=8$ | -3 | $-\frac{1}{32}+\frac{\sqrt{7}}{128}$ | | 3 | $-\frac{53}{1024}-\frac{3\sqrt{31}}{1024}-\frac{5\sqrt{336+82\sqrt{31}}}{1024}$ |
| $\alpha=1.464$ | -2 | $-\frac{3}{128}$ | | 4 | $\frac{51}{2048}+\frac{\sqrt{31}}{2048}$ |
| | -1 | $\frac{9}{32}-\frac{3\sqrt{7}}{128}$ | | 5 | $\frac{7}{1024}+\frac{\sqrt{31}}{1024}+\frac{\sqrt{336+82\sqrt{31}}}{1024}$ |
| | 0 | $\frac{73}{128}$ | | 6 | $-\frac{11}{2048}-\frac{\sqrt{31}}{2048}$ |
| | 1 | $\frac{9}{32}+\frac{3\sqrt{7}}{128}$ | | | |
| | 2 | $-\frac{9}{128}$ | | | |
| | 3 | $-\frac{1}{32}-\frac{\sqrt{7}}{128}$ | | | |
| | 4 | $\frac{3}{128}$ | | | |

**Table 5** Scaling coefficients of coiflets of order $N$, length of filter $2M$ and Sobolev exponent $\alpha$

| | $n$ | $h_n$ |
|---|---|---|
| $N = 4, 2M = 12$ | -5 | $\frac{7}{1024} + \frac{\sqrt{31}}{1024} + \frac{\sqrt{336+82\sqrt{31}}}{2048}$ |
| $\alpha = 2.174$ | -4 | $\frac{7}{2048} - \frac{3\sqrt{31}}{2048}$ |
| | -3 | $-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} - \frac{5\sqrt{336+82\sqrt{31}}}{2048}$ |
| | -2 | $-\frac{39}{2048} + \frac{11\sqrt{31}}{2048}$ |
| | -1 | $\frac{151}{512} + \frac{\sqrt{31}}{512} + \frac{5\sqrt{336+82\sqrt{31}}}{1024}$ |
| | 0 | $\frac{555}{1024} - \frac{7\sqrt{31}}{1024}$ |
| | 1 | $\frac{151}{512} + \frac{\sqrt{31}}{512} - \frac{5\sqrt{336+82\sqrt{31}}}{1024}$ |
| | 2 | $-\frac{47}{1024} + \frac{3\sqrt{31}}{1024}$ |
| | 3 | $-\frac{53}{1024} - \frac{3\sqrt{31}}{1024} + \frac{5\sqrt{336+82\sqrt{31}}}{1024}$ |
| | 4 | $\frac{51}{2048} + \frac{\sqrt{31}}{2048}$ |
| | 5 | $\frac{7}{1024} + \frac{\sqrt{31}}{1024} - \frac{\sqrt{336+82\sqrt{31}}}{1024}$ |
| | 6 | $-\frac{11}{2048} - \frac{\sqrt{31}}{2048}$ |

**Noname manuscript No.**
(will be inserted by the editor)

# Construction of Optimally Conditioned Cubic Spline Wavelets on the Interval

**Dana Černá** · **Václav Finěk**

Received: date / Accepted: date

**Abstract** The paper is concerned with the construction of new spline-wavelet bases on the interval. The resulting bases generate multiresolution analyses on the unit interval with the desired number of vanishing wavelet moments for primal and dual wavelets. Both primal and dual wavelets have compact support. Inner wavelets are translated and dilated versions of well-known wavelets designed by Cohen, Daubechies, and Feauveau. Our objective is to construct interval spline-wavelet bases with the condition number which is close to the condition number of the spline wavelet bases on the real line, especially in the case of the cubic spline wavelets. We show that the constructed set of functions is indeed a Riesz basis for the space $L^2([0,1])$ and for the Sobolev space $H^s([0,1])$ for a certain range of $s$. Then we adapt the primal bases to the homogeneous Dirichlet boundary conditions of the first order and the dual bases to the complementary boundary conditions. Quantitative properties of the constructed bases are presented. Finally, we compare the efficiency of adaptive wavelet scheme for several spline-wavelet bases and we show the superiority of our construction. Numerical examples are presented for one-dimensional and two-dimensional Poisson equations where the solution has steep gradients.

**Keywords** Biorthogonal wavelets · Interval · Spline · Condition number

**Mathematics Subject Classification (2000)** 65T60 · 65N99

## 1 Introduction

Wavelets are by now a widely accepted tool in signal and image processing as well as in numerical simulation. In the field of numerical analysis, methods based on wavelets are successfully used especially for preconditioning of large systems arising from discretization of elliptic partial differential equations, sparse representations of some types of operators and adaptive solving of operator equations. The quantitative performance of such methods strongly depends on the choice of the wavelet basis, in particular on its condition number.

Wavelet bases on a bounded domain are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by tensor product technique to the $n$-dimensional cube. Finally by splitting the domain into subdomains which are images of $(0,1)^n$ under appropriate parametric mappings one can obtain wavelet bases on a fairly general domain. Thus, the properties of the employed wavelet basis on the interval are crucial for the properties of the resulting bases on general domain.

D. Černá
Department of Mathematics and Didactics of Mathematics, Technical University in Liberec,
Studentská 2, 461 17 Liberec, Czech Republic
Tel.: +420-48-535-2407
Fax: +420-48-535-2332
E-mail: dana.cerna@tul.cz

V. Finěk
Department of Mathematics and Didactics of Mathematics, Technical University in Liberec,
Studentská 2, 461 17 Liberec, Czech Republic

Biorthogonal spline-wavelet bases on the unit interval were constructed in [16]. The disadvantage of them is their bad condition which causes problems in practical applications. Some modifications which lead to better conditioned bases were proposed in [2], [17], [24], and [33]. The recent construction by M. Primbs, see [12], [24], or [25], seems to outperform the previous constructions with respect to the Riesz bounds as well as spectral properties of the corresponding stiffness matrices in the case of linear and quadratic spline-wavelets. In this paper, we focus on cubic spline wavelets and we construct interval spline-wavelet bases with the condition number which is close to the condition number of the spline wavelet bases on the real line. It is known that the condition number of the wavelet basis on the real line is less or equal to the condition number of the interval wavelet basis, where the inner functions are restrictions of scaling functions and wavelets on the real line.

First of all, we summarize the desired properties:

- *Riesz basis property.* The functions form a Riesz basis of the space $L^2([0,1])$.
- *Locality.* The basis functions are local. Then the corresponding decomposition and reconstruction algorithms are simple and fast.
- *Biorthogonality.* The primal and dual wavelet bases form a biorthogonal pair.
- *Polymial exactness.* The primal MRA has polynomial exactness of order $N$ and the dual MRA has polynomial exactness of order $\tilde{N}$. As in [9], $N+\tilde{N}$ has to be even and $\tilde{N} \geq N$.
- *Smoothness.* The smoothness of primal and dual wavelet bases is another desired property. It ensures the validity of norm equivalences, for details see below.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form. It is desirable property for the fast computation of integrals involving primal scaling functions and wavelets.
- *Well-conditioned bases.* Our objective is to construct wavelet bases with improved condition number, especially for larger values of $N$ and $\tilde{N}$.

From the viewpoint of numerical stability, ideal wavelet bases are orthogonal wavelet bases. However, they are usually avoided in numerical treatment of partial differential and integral equations, because they are not accessible analytically, the complementary boundary conditions can not be satisfied and it is not possible to increase the number of vanishing wavelet moments independent from the order of accuracy. Moreover, sufficiently smooth orthogonal wavelets typically have a large support.

Biorthogonal wavelet bases on the unit interval derived from B-splines were constructed also in [8] and [19] and they were adapted to homogeneous Dirichlet boundary conditions in [20]. These bases are well-conditioned, but have globally supported dual basis functions. Another construction of spline-wavelets was proposed in [4], but the corresponding dual bases are unknown so far. We should also mention the construction of spline multiwavelets [15], [22], and [28], though the dual wavelets have a low Sobolev regularity.

The paper is organized as follows. Section 2 provides a short introduction to the concept of wavelet bases. Section 3 is concerned with the construction of primal multiresolution analysis on the interval. The primal scaling functions are B-splines defined on the Schoenberg sequence of knots, which have been used also in [4], [8], and [24]. In Section 4 we construct dual multiresolution analysis. There are two types of boundary scaling functions. The functions of the first type are defined in order to preserve the full degree of polynomial exactness as in [1] and [10]. The construction of the scaling functions of the second type is a delicate task, because the low condition number and nestedness of the multiresolution spaces have to be preserved. Section 5 is concerned with the computation of refinement matrices. In Section 6 wavelets are constructed by the method of stable completion proposed in [18]. The construction of initial stable completion is along the lines of [16]. In Section 7 we show that the constructed set of functions is indeed a Riesz basis for the space $L^2([0,1])$ and for the Sobolev space $H^s([0,1])$ for a certain range of $s$. In Section 8 we adapt the primal bases to the homogeneous Dirichlet boundary conditions of the first order and the dual bases to the complementary boundary conditions. Quantitative properties of the constructed bases are presented in Section 9. Finally, in Section 10, we compare the efficiency of adaptive wavelet scheme for several spline-wavelet bases and we show the superiority of our construction. Numerical examples are presented for one-dimensional and two-dimensional Poisson equations where the solution has steep gradients.

## 2 Wavelet bases

This section provides a short introduction to the concept of wavelet bases. Let us introduce some notation. We use $\mathbb{N}$, $\mathbb{Z}$, $\mathbb{Q}$, and $\mathbb{R}$ to denote the set of positive integers, integers, rational numbers, and real numbers, respectively. Let $\mathbb{N}_{j_0}$ denote the set of integers which are greater than or equal to $j_0$.

We consider the domain $\Omega \subset \mathbb{R}^d$ and the space $L^2(\Omega)$ with inner product $\langle \cdot, \cdot \rangle$ and induced norm $\|\cdot\|$. Let $\mathscr{J}$ be some index set and let each index $\lambda \in \mathscr{J}$ take the form $\lambda = (j,k)$, where $|\lambda| = j \in \mathbb{Z}$ is *scale* or *level*. Let $l^2(\mathscr{J})$ be a space of all sequences $b = \{b_\lambda\}_{\lambda \in \mathscr{J}}$ such that

$$\|b\|_{l_2(\mathscr{J})} := \left( \sum_{\lambda \in \mathscr{J}} |b_\lambda|^2 \right)^{\frac{1}{2}} < \infty. \tag{1}$$

**Definition 1.** A family $\Psi := \{\psi_\lambda \in \mathscr{J}\} \subset L^2(\Omega)$ is called a *wavelet basis* of $L^2(\Omega)$, if

*i*) $\Psi$ is a *Riesz basis* for $L^2(\Omega)$, it means that the linear span of $\Psi$ is dense in $L^2(\Omega)$ and there exist constants $c, C \in (0, \infty)$ such that

$$c\,\|\mathrm{b}\|_{l_2(\mathscr{J})} \leq \left\| \sum_{\lambda \in \mathscr{J}} b_\lambda \psi_\lambda \right\| \leq C\,\|\mathrm{b}\|_{l_2(\mathscr{J})} \quad \text{for all} \quad \mathrm{b} = \{b_\lambda\}_{\lambda \in \mathscr{J}} \in l^2(\mathscr{J}). \tag{2}$$

Constants $c_\Psi := \sup\{c : c \text{ satisfies } (2)\}, C_\Psi := \inf\{C : C \text{ satisfies } (2)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\Psi / c_\Psi$ is called the *condition number* of $\Psi$.

*ii*) The functions are *local* in the sense that

$$\operatorname{diam}(\Omega_\lambda) \leq \mathrm{C}2^{-|\lambda|} \quad \text{for all} \quad \lambda \in \mathscr{J}, \tag{3}$$

where $\Omega_\lambda$ is the support of $\psi_\lambda$, and at a given level $j$ the supports of only finitely many wavelets overlap in any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi} = \left\{\tilde{\psi}_\lambda, \lambda \in \tilde{\mathscr{J}}\right\} \subset L^2(\Omega)$ biorthogonal to $\Psi$, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle = \delta_{i,j}\delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathscr{J}, \quad (j,l) \in \tilde{\mathscr{J}}. \tag{4}$$

Here, $\delta_{i,j}$ denotes the Kronecker delta, i.e. $\delta_{i,i} := 1$, $\delta_{i,j} := 0$ for $i \neq j$. This family is also a Riesz basis for $L^2(\Omega)$. The basis $\Psi$ is called *primal* wavelet basis, $\tilde{\Psi}$ is called *dual* wavelet basis.

In many cases, the wavelet system $\Psi$ is constructed with the aid of a multiresolution analysis.

**Definition 2.** A sequence $S = \{S_j\}_{j \in \mathbb{N}_{j_0}}$ of closed linear subspaces $S_j \subset L^2(\Omega)$ is called a *multiresolution* or *multiscale analysis*, if

$$S_{j_0} \subset S_{j_0+1} \subset \ldots \subset S_j \subset S_{j+1} \subset \ldots L^2(\Omega) \quad \text{and} \quad \overline{\left(\cup_{j \in \mathbb{N}_{j_0}} S_j\right)} = L^2(\Omega). \tag{5}$$

The nestedness of the multiresolution analysis implies the existence of the *complement spaces* $W_j$ such that

$$S_{j+1} = S_j \oplus W_j, \tag{6}$$

where $\oplus$ denotes the direct product.

We now assume that $S_j$ and $W_j$ are spanned by sets of basis functions

$$\Phi_j := \left\{\phi_{j,k}, k \in \mathscr{I}_j\right\}, \quad \Psi_j := \left\{\psi_{j,k}, k \in \mathscr{J}_j\right\}, \tag{7}$$

where $\mathscr{I}_j$, $\mathscr{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis is given by

$$\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j \tag{8}$$

and the overall wavelet basis of $L^2(\Omega)$ is obtained by

$$\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j. \tag{9}$$

The single-scale and the multiscale bases are interrelated by the *wavelet transform* $\mathbf{T}_{j,s} : l^2(I_{j+s}) \to l^2(I_{j+s})$,

$$\Psi_{j,s} = \mathbf{T}_{j,s} \Phi_{j+s}. \tag{10}$$

The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{S}$ with a dual scaling basis $\tilde{\Phi}$.

*Polynomial exactness* of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\Pi_{N-1}(\Omega) \subset S_j, \quad \Pi_{\tilde{N}-1}(\Omega) \subset \tilde{S}_j, \quad j \geq j_0, \tag{11}$$

where $\Pi_m(\Omega)$ is the space of all algebraic polynomials on $\Omega$ of degree at most $m$.

## 3 Primal Scaling Basis

The primal scaling bases will be the same as bases designed by Chui and Quak in [8], because they are known to be well-conditioned. A big advantage of this approach is that it readily adapts to the bounded interval by introducing multiple knots at the endpoints. Let $N$ be the desired order of polynomial exactness of primal scaling basis and let $\mathbf{t}^j = \left(t_k^j\right)_{k=-N+1}^{2^j+N-1}$ be a *Schoenberg sequence of knots* defined by

$$\begin{aligned} t_k^j &:= 0, \qquad k = -N+1, \ldots, 0, \\ t_k^j &:= \frac{k}{2^j}, \quad k = 1, \ldots, 2^j - 1, \\ t_k^j &:= 1, \qquad k = 2^j, \ldots, 2^j + N - 1. \end{aligned} \tag{12}$$

The corresponding *B-splines of order N* are defined by

$$B_{k,N}^j(x) := \left(t_{k+N}^j - t_k^j\right)\left[t_k^j, \ldots, t_{k+N}^j\right](t-x)_+^{N-1}, \quad x \in \langle 0, 1 \rangle, \tag{13}$$

where $(x)_+ := \max\{0, x\}$. The symbol $[t_k, \ldots t_{k+N}] f$ is the $N$-th divided difference of $f$ which is recursively defined as

$$\begin{aligned} [t_k, \ldots, t_{k+N}] f &= \frac{[t_{k+1}, \ldots, t_{k+N}] f - [t_k, \ldots, t_{k+N-1}] f}{t_{k+N} - t_k} \quad \text{if} \quad t_k \neq t_{k+N}, \\ &= \frac{f^{(N)}(t_k)}{N!} \qquad\qquad\qquad\qquad \text{if} \quad t_k = t_{k+N}, \end{aligned}$$

with $[t_k] f = f(t_k)$.

The set $\Phi_j = \left\{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\right\}$ of primal scaling functions is then simply defined by

$$\phi_{j,k} = 2^{j/2} B_{k,N}^j, \quad k = -N+1, \ldots, 2^j - 1, \quad j \geq 0. \tag{14}$$

Thus there are $2^j - N + 1$ inner scaling functions and $N - 1$ functions at each boundary. Figure 1 shows the primal scaling functions for $N = 4$ and $j = 3$. Inner scaling functions are translations and dilations of one function $\phi$ which corresponds to the primal scaling function constructed by Cohen, Daubechies, Feauveau in [9]. In the following, we consider $\phi$ from [9] which is shifted so that its support is $[0, N]$.

We define the primal multiresolution spaces by

$$S_j = \operatorname{span} \Phi_j. \tag{15}$$

**Lemma 3.** *Under the above assumptions, the following holds:*

*i) For any $j_0 \in \mathbb{N}$ the sequence $\mathscr{S} = \left\{S_j\right\}_{j \geq j_0}$ forms a multiresolution analysis of $L^2([0,1])$.*

*ii) The spaces $S_j$ are exact of order $N$, i.e.*

$$\Pi_{N-1}([0,1]) \subset S_j, \quad j \geq 1. \tag{16}$$

The proof can be found in [8], [24], [29].

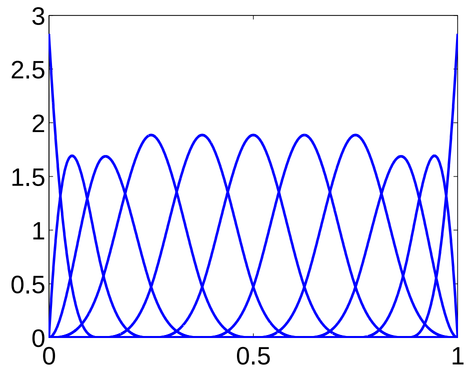



**Fig. 1** Primal scaling functions for $N = 4$ and $j = 3$ without boundary conditions.

## 4 Dual Scaling Basis

The desired property of dual scaling basis $\tilde{\Phi}$ is the biorthogonality to $\Phi$ and the polynomial exactness of order $\tilde{N}$. Let $\tilde{\phi}$ be the dual scaling function which was designed by Cohen, Daubechies, and Feauveau in [9] and which is shifted so that $\langle \phi, \tilde{\phi} \rangle = 0$, i.e. its support is $\left[-\tilde{N}+1, N+\tilde{N}-1\right]$. In this case $\tilde{N} \geq N$ and $\tilde{N}+N$ has to be an even number. It is known that there exist sequences $\{h_k\}_{k\in\mathbb{Z}}$ and $\{\tilde{h}_k\}_{k\in\mathbb{Z}}$ such that the functions $\phi$ and $\tilde{\phi}$ satisfy the *refinement equations*

$$\phi(x) = \sum_{k\in\mathbb{Z}} h_k \phi(2x-k), \quad \tilde{\phi}(x) = \sum_{k\in\mathbb{Z}} \tilde{h}_k \tilde{\phi}(2x-k), \quad x \in \mathbb{R}. \tag{17}$$

The parameters $h_k$ and $\tilde{h}_k$ are called *scaling coefficients*. By biorthogonality of $\phi$ and $\tilde{\phi}$, we have

$$2\sum_{k\in\mathbb{Z}} h_{2m+k}\tilde{h}_k = \delta_{0,m}, \quad m \in \mathbb{Z}. \tag{18}$$

Note that only coefficients $h_0, \ldots, h_N$ and $\tilde{h}_{-\tilde{N}+1}, \ldots, \tilde{h}_{N+\tilde{N}-1}$ may be nonzero.

In the sequel, we assume that

$$j \geq j_0 := \left\lceil \log_2\left(N+2\tilde{N}-3\right)\right\rceil \tag{19}$$

so that the supports of the boundary functions are contained in $[0,1]$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} = 2^{j/2}\tilde{\phi}\left(2^j\cdot -k\right), \quad k = \tilde{N}-1, \ldots, 2^j-N-\tilde{N}+1. \tag{20}$$

There will be two types of basis functions at each boundary. In the following, it will be convenient to abbreviate the boundary and inner index sets by

$$\mathcal{I}_j^{L,1} = \left\{-N+1, \ldots, -N+\tilde{N}\right\}, \tag{21}$$
$$\mathcal{I}_j^{L,2} = \left\{-N+\tilde{N}+1, \ldots, \tilde{N}-2\right\}, \tag{22}$$
$$\mathcal{I}_j^{0} = \left\{\tilde{N}-1, \ldots, 2^j-N-\tilde{N}+1\right\}, \tag{23}$$
$$\mathcal{I}_j^{R,2} = \left\{2^j-N-\tilde{N}+2, \ldots 2^j-\tilde{N}-1\right\}, \tag{24}$$
$$\mathcal{I}_j^{R,1} = \left\{2^j-\tilde{N}, \ldots, 2^j-1\right\}, \tag{25}$$

and

$$\mathcal{I}_j^{L} = \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} = \left\{-N+1, \ldots, \tilde{N}-2\right\}, \tag{26}$$
$$\mathcal{I}_j^{R} = \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{2^j-N-\tilde{N}+2, \ldots, 2^j-1\right\}, \tag{27}$$
$$\mathcal{I}_j = \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^{0} \cup \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{-N+1, \ldots, 2^j-1\right\}. \tag{28}$$

Basis functions of the first type are defined to preserve polynomial exactness by the same way as in [1], [10] :

$$\theta_{j,k} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \langle p_{k+N-1}, \phi(\cdot - l)\rangle \tilde{\phi}\left(2^j \cdot -l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,1}, \tag{29}$$

where $\left\{p_0, \ldots, p_{\tilde{N}-1}\right\}$ is a basis of $\Pi_{\tilde{N}-1}([0,1])$. In Lemma 6 we show that the resulting dual scaling functions do not depend on the choice of the polynomial basis. In our case, $p_k$ are the Bernstein polynomials defined by

$$p_k(x) := b^{-\tilde{N}+1} \binom{\tilde{N}-1}{k} x^k (b-x)^{\tilde{N}-1-k}, \quad k = 0, \ldots, \tilde{N}-1, \quad x \in \mathbb{R}. \tag{30}$$

The Bernstein polynomials were used also in [16]. On the contrary to [16], in our case the choice of polynomials does not affect the resulting dual scaling basis $\tilde{\Psi}$, but it has only the effect of stabilization of the computation, for details see Lemma 6 and the discussion below.

The definition of basis functions of the second type is a delicate task, because the low condition number and the nestedness of the multiresolution spaces have to be preserved. This means that the relation $\theta_{j,k} \subset \tilde{V}_j \subset \tilde{V}_{j+1}$, $k \in \mathscr{I}_j^{L,2}$, should hold. Therefore we define $\theta_{j,k}$, $k \in \mathscr{I}_j^{L,2}$, as linear combinations of functions which are already in $\tilde{V}_{j+1}$. To obtain well-conditioned bases, we define functions $\theta_{j,k}$, $k \in \mathscr{I}_j^{L,2}$, which are close to $\tilde{\phi}_{j,k}^{\mathbb{R}} := 2^{j/2}\tilde{\phi}\left(2^j \cdot -k\right)$, because $\tilde{\phi}_{j,k}^{\mathbb{R}}$, $k \in \mathscr{I}_j^{L,2}$, are biorthogonal to the inner primal scaling functions and the condition of $\left\{\tilde{\phi}_{j,k}^{\mathbb{R}}, k \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^{0}\right\}$ is the same as the condition of the set of inner dual basis functions.

For this reason, we define the basis functions of the second type by

$$\theta_{j,k} = 2^{\frac{j}{2}} \sum_{l=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot -2k - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,2}, \tag{31}$$

where $\tilde{h}_i$ are the scaling coefficients corresponding to the scaling function $\tilde{\phi}$. Then $\theta_{j,k}$ is close to $\tilde{\phi}_{j,k}^{\mathbb{R}}|_{[0,1]}$, because by (17) we have

$$\tilde{\phi}_{j,k}^{\mathbb{R}}|_{[0,1]} = 2^{\frac{j}{2}} \sum_{k=-\tilde{N}+1}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}\left(2^{j+1} \cdot -2k - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,2}. \tag{32}$$

Figure 2 shows the functions $\theta_{j,k}$ and $\tilde{\phi}_{j,k}^{\mathbb{R}}$ for $N = 4$, $\tilde{N} = 6$, and $j = 4$.

The boundary functions at the right boundary are defined to be symmetric with the left boundary functions:

$$\theta_{j,k} = \theta_{j,2^j-k}(1 - \cdot), \quad k \in \mathscr{I}_j^R. \tag{33}$$

It is easy to see that

$$\theta_{j+1,k} = 2^{1/2}\theta_{j,k}(2\cdot), \quad k \in \mathscr{I}_j^L \tag{34}$$

for left boundary functions and

$$\theta_{j+1,k}(1 - \cdot) = 2^{1/2}\theta_{j,k}(1 - 2\cdot), \quad k \in \mathscr{I}_j^R \tag{35}$$

for right boundary functions.

Since the set $\Theta_j := \left\{\theta_{j,k}, k \in \mathscr{I}_j\right\}$ is not biorthogonal to $\Phi_j$, we derive a new set

$$\tilde{\Phi}_j := \left\{\tilde{\phi}_{j,k}, k \in \mathscr{I}_j\right\} \tag{36}$$

from $\Theta_j$ by biorthogonalization. Let

$$\mathbf{Q}_j = \left(\langle \phi_{j,k}, \theta_{j,l}\rangle\right)_{k,l \in \mathscr{I}_j}. \tag{37}$$

Then viewing $\tilde{\Phi}_j$ and $\Theta_j$ as column vectors we define

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T}\Theta_j, \tag{38}$$

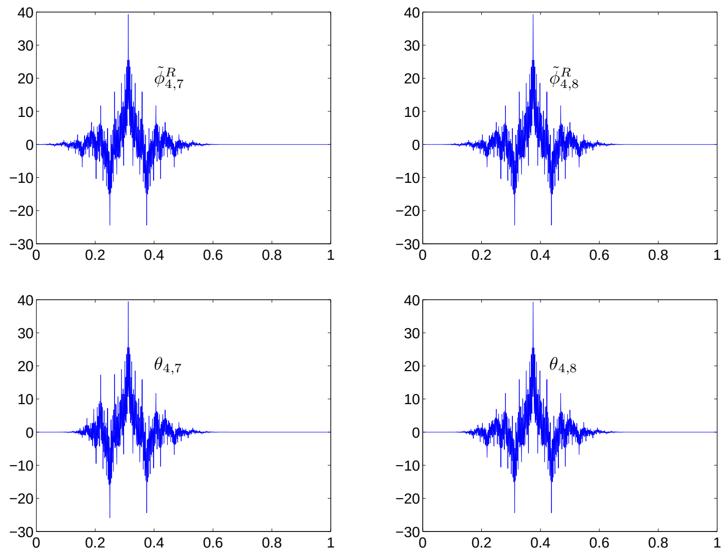



**Fig. 2** The functions $\tilde{\phi}^{\mathbb{R}}_{4,k}$ and $\theta_{4,k}$ for $N = 4$ and $\tilde{N} = 6$.

assuming that $\mathbf{Q}_j$ is invertible, which is the case of all choices of $N$ and $\tilde{N}$ considered in our numerical examples below.

Then $\tilde{\Phi}_j$ is biorthogonal to $\Phi_j$, because

$$\left\langle \Phi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \mathbf{Q}_j^{-T} \Theta_j \right\rangle = \mathbf{Q}_j \mathbf{Q}_j^{-1} = \mathbf{I}_{\#\mathscr{I}_j}, \tag{39}$$

where the symbol # denotes the cardinality of the set and $\mathbf{I}_m$ denotes the identity matrix of the size $m \times m$.

**Lemma 4.** *i) Let $\Phi_j$, $\Theta_j$ be defined as above. Then the matrices*

$$\mathbf{Q}_{j,L} = \left( \langle \phi_{j,k}, \theta_{j,l} \rangle \right)_{k,l \in \mathscr{I}_j^L} \quad \text{and} \quad \mathbf{Q}_{j,R} = \left( \langle \phi_{j,k}, \theta_{j,l} \rangle \right)_{k,l \in \mathscr{I}_j^R} \tag{40}$$

*are independent of $j$, i.e. there are matrices $\mathbf{Q}_L$, $\mathbf{Q}_R$ such that*

$$\mathbf{Q}_{j,L} = \mathbf{Q}_L, \quad \mathbf{Q}_{j,R} = \mathbf{Q}_R. \tag{41}$$

*Moreover, the matrix $\mathbf{Q}_R$ results from the matrix $\mathbf{Q}_L$ by reversing the ordering of rows and columns, which means that*

$$(\mathbf{Q}_R)_{k,l} = (\mathbf{Q}_L)_{2^j - N - k, 2^j - N - l}, \quad k,l \in \mathscr{I}_j^R. \tag{42}$$

*ii) The following holds:*

$$(\mathbf{Q}_j)_{k,l} = \delta_{k,l}, \quad k \in \mathscr{I}_j, \; l \in \mathscr{I}_j^0. \tag{43}$$

*iii) The following holds:*

$$(\mathbf{Q}_j)_{k,l} = 0, \quad k \in \mathscr{I}_j^0, \; l \in \mathscr{I}_j^L \cup \mathscr{I}_j^R. \tag{44}$$

*Proof* Due to (34) and by substitution we have for $k,l \in \mathscr{I}_j^L$

$$\langle \phi_{j,k}, \theta_{j,l} \rangle = \left\langle 2^{\frac{j-j_0}{2}} \phi_{j_0,k} \left( 2^{j-j_0} \cdot \right), 2^{\frac{j-j_0}{2}} \theta_{j_0,l} \left( 2^{j-j_0} \cdot \right) \right\rangle = \langle \phi_{j_0,k}, \theta_{j_0,l} \rangle. \tag{45}$$

Therefore, $\mathbf{Q}_{j,L} = \mathbf{Q}_{j_0,L} = \mathbf{Q}_L$, i.e. the matrix $\mathbf{Q}_{j,L}$ is independent of $j$. Due to (35) $\mathbf{Q}_{j,R}$ is independent of $j$ too. The property (42) is a direct consequence of the reflection invariance (33).

The property $ii)$ follows from the biorthogonality of $\{\phi(\cdot - k)\}_{k\in\mathbb{Z}}$ and $\{\tilde{\phi}(\cdot - l)\}_{l\in\mathbb{Z}}$. It also implies (44) for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{R,1}$. It remains to prove (44) for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^{R,2}$. By the definition of dual scaling functions of the second type (31), refinement relation (17) for the dual scaling function $\tilde{\phi}$, and (18), we have for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{L,2}$,

$$\langle \phi_{j,k}, \theta_{j,l} \rangle = \left\langle \phi(\cdot - k), \sqrt{2} \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_l \tilde{\phi}(2\cdot - 2l - m)\,|_{[0,1]} \right\rangle \tag{46}$$

$$= 2 \left\langle \sum_{n=0}^{N} h_n \phi(2\cdot - 2k - n), \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} \tilde{h}_m \tilde{\phi}(2\cdot - 2l - m)\,|_{[0,1]} \right\rangle \tag{47}$$

$$= 2 \sum_{n=0}^{N} \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_n \tilde{h}_m \delta_{2k+n,2l+m} = 2 \sum_{m=\tilde{N}-1-2k}^{N+\tilde{N}-1} h_{2l-2k+m} \tilde{h}_m \tag{48}$$

$$= 2 \sum_{m\in\mathbb{Z}} h_{2l-2k+m} \tilde{h}_m = 0. \tag{49}$$

By (33), the relation (44) holds also for $k \in \mathscr{I}_j^0$, $l \in \mathscr{I}_j^{R,2}$.

Thus, we can write

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T} \Theta_j = \begin{pmatrix} \mathbf{Q}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{Q}_R \end{pmatrix}^{-T} \Theta_j = \begin{pmatrix} \mathbf{Q}_L^{-T} & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{Q}_R^{-T} \end{pmatrix} \Theta_j, \tag{50}$$

Since the matrix $\mathbf{Q}_j$ is symmetric in the sense of (42), the properties (33), (34), and (35) hold for $\tilde{\phi}_{j,k}$ as well.

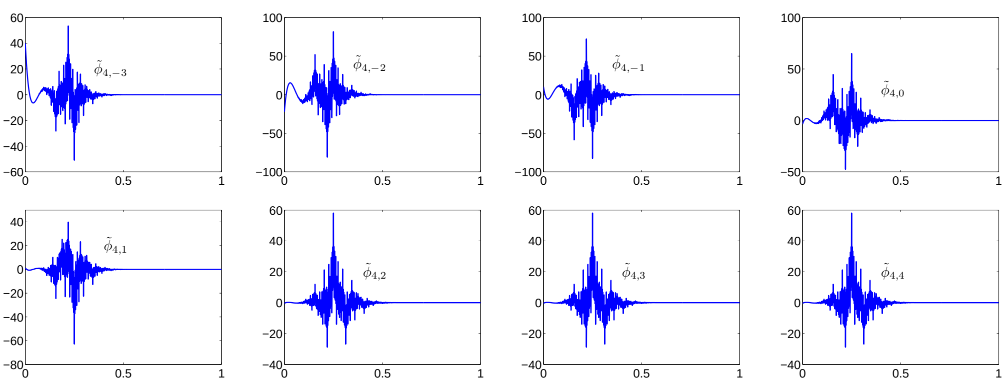



**Fig. 3** Boundary dual scaling functions for $N = 4$ and $\tilde{N} = 6$ without boundary conditions.

**Remark 5.** *It is known that the scaling function $\tilde{\phi}$ has typically a low Sobolev regularity for smaller values of $\tilde{N}$. Thus the functions $\theta_{j,k}$ have a low Sobolev regularity for smaller values of $\tilde{N}$, too. Therefore, we do not obtain the sufficiently accurate entries of the matrix $\mathbf{Q}_j$ directly by classical quadratures. Fortunately, we are able to compute the matrix $\mathbf{Q}_j$ precisely up to the round off errors. For $k \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2}$, $l \in \mathscr{I}_j^{L,1}$ we have*

$$\langle \phi_{j,k}, \theta_{j,l} \rangle = \sum_{m=-N-\tilde{N}+2}^{\tilde{N}-2} \sum_{n=0}^{\tilde{N}-1} c_{l,n} \langle (\cdot)^n, \phi(\cdot - m) \rangle \langle \phi(\cdot - k), \tilde{\phi}(\cdot - m) \rangle_{L^2(\langle 0,1 \rangle)}, \tag{51}$$

*with $c_{l,n}$ given by (63). Since $\phi$ is a piecewise polynomial function and $\tilde{\phi}$ is refinable, for $k \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^{L,2}$, $l \in \mathscr{I}_j^{L,1}$ we can compute the entries of $\mathbf{Q}_j$ by the method from [11]. By refinement relation we easily obtain the following relations for the computation of the remaining entries of $\mathbf{Q}^L$:*

$$\begin{aligned}\left\langle \phi_{j,k}, \theta_{j,l} \right\rangle &= \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} \tilde{h}_m \left\langle \phi_{0,k}, \tilde{\phi}\left(\cdot - 2k - m\right) \right\rangle, \quad k = -N+1, \ldots, -1,\ l \in \mathscr{I}_j^{L,2}, \\ &= 2^{-1} \sum_{m=\tilde{N}-1-2l}^{N+\tilde{N}-1} h_{2k-2l+m} \tilde{h}_m, \qquad k = 0, \ldots, \tilde{N}-2,\ l \in \mathscr{I}_j^{L,2}.\end{aligned}$$

*Since the submatrix $\mathbf{Q}_R$ is obtained from a matrix $\mathbf{Q}_L$ by reversing the ordering of rows and columns, the matrix $\mathbf{Q}_j$ can be indeed computed precisely up to the round off errors.*

Now we show that the resulting dual scaling basis $\tilde{\Phi}$ does not depend on the choice of polynomial basis of the space $\Pi_{\tilde{N}}\left([0,1]\right)$ in the formula (29).

**Lemma 6.** *We suppose that $P^1 = \left\{ p_0^1, \ldots, p_{\tilde{N}-1}^1 \right\}$, $P^2 = \left\{ p_0^2, \ldots, p_{\tilde{N}-1}^2 \right\}$ are two different bases of the space $\Pi_{\tilde{N}}\left([0,1]\right)$ and for $i = 1, 2$ we define the sets $\Theta_j^i = \left\{ \theta_{j,k}^i \right\}_{k=-N+1}^{2^j-1}$ by*

$$\begin{aligned}\theta_{j,k}^i &= 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle p_{k+N-1}^i, \phi\left(\cdot - l\right) \right\rangle \tilde{\phi}\left(2^j \cdot - l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,1}, \\ \theta_{j,k}^i &= \theta_{j,2^j-N-k}^i, \qquad k \in \mathscr{I}_j^{R,1}, \\ \theta_{j,k}^i &= \theta_{j,k}, \qquad k \in \mathscr{I}_j^{L,2} \cup \mathscr{I}_j^0 \cup \mathscr{I}_j^{R,2}.\end{aligned}$$

*Furthermore, we define*

$$\mathbf{Q}_j^i = \left\langle \Phi_j, \Theta_j^i \right\rangle, \quad \tilde{\Phi}_j^i = \left(\mathbf{Q}_j^i\right)^{-T} \Theta_j^i, \quad i = 1, 2, \tag{52}$$

*and we assume that $\mathbf{Q}_j^i$ is nonsingular. Then $\tilde{\Phi}_j^1 = \tilde{\Phi}_j^2$.*

*Proof* Since $P^1$ and $P^2$ are both bases of the space $\Pi_{\tilde{N}}\left([0,1]\right)$, there exists a regular matrix $\mathbf{B}_L$ such that $P^2 = \mathbf{B}_L P^1$. The consequence is that

$$\Theta^2 = \mathbf{B}_j \Theta^1, \tag{53}$$

with

$$\mathbf{B}_j = \begin{pmatrix} \mathbf{B}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_j^0} & \\ & & \mathbf{B}_R \end{pmatrix}, \tag{54}$$

where $\mathbf{B}_R$ results from a matrix $\mathbf{B}_L$ by reversing the ordering of rows and columns, which means that

$$\left(\mathbf{B}_R\right)_{k,l} = \left(\mathbf{B}_L\right)_{2^j-N-k, 2^j-N-l}, \quad k, l \in \mathscr{I}_j^{L,1}. \tag{55}$$

Therefore, we have

$$\tilde{\Phi}_j^2 = \left(\mathbf{Q}_j^2\right)^{-T} \Theta_j^2 = \left(\mathbf{Q}_j^1\right)^{-T} \mathbf{B}_j^{-1} \mathbf{B}_j \Theta_j^1 = \tilde{\Phi}_j^1. \tag{56}$$

Although the choice of polynomial basis does not influence the resulting dual scaling basis, it has an influence on the stability of the computation and the preciseness of the results, because some choices of the polynomial bases leads to the critical values of the condition number of the biorthogonalization matrix. We present the condition numbers of the matrix $\mathbf{Q}_L$ for the monomial basis $\left\{1, x, x^2, \ldots x^{\tilde{N}-1}\right\}$ and Bernstein polynomials (30) with the parameters $b = 4$ and $b = 10$ in Table 4. In our numerical experiments in Section 9 we choose $b = 10$.

**Remark 7.** *In the case of linear primal basis, i.e. $N = 2$, there are no boundary dual functions of the second type. In [24] the primal scaling functions and the inner dual scaling functions are the same as ours. The boundary dual functions before biorthogonalizations are defined by (29) with the same choice of polynomials $p_0, \ldots, p_{\tilde{N}-1}$ as in [10]. Due to the Lemma 6, for $N = 2$ the wavelet basis in [24] is identical to the wavelet basis constructed in this section.*

For the proof of Theorem 9 below and also for deriving of refinement matrices we will need the following lemma.

**Lemma 8.** *For the left boundary functions of the first type there exist refinement coefficients* $m_{n,k}$, $k \in \mathscr{I}_j^{L,1}$, $n \in \mathscr{I}_j^{L,1} \cup \mathscr{I}_j^3$, $\mathscr{I}_j^3 := \left\{\tilde{N}-1,\ldots,3\tilde{N}+N-5\right\}$ *such that*

$$\theta_{j,k} = \sum_{n=-N+1}^{-N+\tilde{N}} m_{n,k}\theta_{j+1,n} + \sum_{n=\tilde{N}-1}^{3\tilde{N}+N-5} m_{n,k}\theta_{j+1,n}, \quad k \in \mathscr{I}_j^{L,1}. \tag{57}$$

*Proof* Let $\Theta_j^0 = \left\{\theta_{j,k}, k \in \mathscr{I}_j^3\right\}$ and $\Theta_j^{L,1,mon} = \left\{\theta_{j,k}^{mon}, k \in \mathscr{I}_j^{L,1}\right\}$ be defined by

$$\theta_{j,k}^{mon} = 2^{j/2} \sum_{l=-N-\tilde{N}+2}^{\tilde{N}-2} \left\langle (\cdot)^i, \phi(\cdot - l)\right\rangle \tilde{\phi}\left(2^j \cdot -l\right)|_{[0,1]}, \quad k \in \mathscr{I}_j^{L,1}. \tag{58}$$

Then

$$\Theta_j^{L,1,mon} = \left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \Theta_{j+1}^{L,1,mon} \\ \Theta_{j+1}^0 \end{pmatrix}, \tag{59}$$

where the refinement matrix $\mathbf{M}^{mon} = \left\{m_{n,k}^{mon}\right\}_{n\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^3, k\in\mathscr{I}_j^{L,1}}$ is given by

$$m_{n,k}^{mon} = \frac{1}{\sqrt{2}} 2^{-k}, \quad k = n,\ n \in \mathscr{I}_j^{L,1}, \tag{60}$$

$$= \frac{1}{\sqrt{2}} \sum_{q=\left\lceil \frac{n-N-\tilde{N}+1}{2}\right\rceil}^{\tilde{N}-2} \left\langle (\cdot)^{k+N-1}, \phi(\cdot - q)\right\rangle \tilde{h}_{n-2q}, \quad k \in \mathscr{I}_j^{L,1},\ n \in \mathscr{I}_j^3, \tag{61}$$

$$= 0, \quad \text{otherwise.} \tag{62}$$

For deriving of $\mathbf{M}^{mon}$ see [16]. It is known that the coefficients of Bernstein polynomials in a monomial basis are given by

$$c_{l,n} = (-1)^{l-n} \binom{\tilde{N}-1}{n} \binom{n}{l} b^{-n}, \quad \text{if} \quad n \geq l, \tag{63}$$

$$= 0, \qquad \text{otherwise.} \tag{64}$$

Hence, the matrix $\mathbf{C} = \left\{C_{l,n}\right\}_{l,n=-N+1}^{-N+\tilde{N}}$ is an upper triangular matrix with nonzero entries on the diagonal which implies that $\mathbf{C}$ is invertible. We denote $\Theta_j^{L,1} = \left\{\theta_{j,k}, k \in \mathscr{I}_j^{L,1}\right\}$ and we obtain

$$\Theta_j^{L,1} = \mathbf{C}\Theta_j^{L,1,mon} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \Theta_{j+1}^{L,1,mon} \\ \Theta_{j+1}^0 \end{pmatrix} = \mathbf{C}\left(\mathbf{M}^{mon}\right)^T \begin{pmatrix} \mathbf{C}^{-1} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \begin{pmatrix} \Theta_{j+1}^{L,1} \\ \Theta_{j+1}^0 \end{pmatrix}. \tag{65}$$

**Table 1** Condition numbers of the matrices $\mathbf{Q}_L$

| $N$ | $\tilde{N}$ | mon. | $b=4$ | $b=10$ | $N$ | $\tilde{N}$ | mon. | $b=4$ | $b=10$ |
|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 6.68e+00 | 9.94e+00 | 3.16e+01 | 4 | 4 | 2.46e+04 | 6.75e+02 | 1.33e+04 |
| 2 | 4 | 4.66e+02 | 1.94e+01 | 9.48e+02 | 4 | 6 | 1.30e+07 | 2.94e+04 | 7.34e+04 |
| 2 | 6 | 1.40e+05 | 1.00e+02 | 4.47e+03 | 4 | 8 | 1.24e+10 | 6.24e+06 | 9.42e+04 |
| 2 | 8 | 1.03e+08 | 8.52e+03 | 5.81e+03 | 4 | 10 | 1.92e+13 | 2.26e+09 | 5.24e+04 |
| 2 | 10 | 1.48e+11 | 1.67e+06 | 1.58e+03 | 5 | 5 | 5.34e+06 | 3.29e+04 | 1.26e+05 |
| 3 | 3 | 2.18e+02 | 1.07e+02 | 1.00e+03 | 5 | 7 | 5.62e+09 | 6.91e+06 | 3.73e+05 |
| 3 | 5 | 3.73e+04 | 1.88e+02 | 1.05e+04 | 5 | 9 | 9.39e+12 | 2.57e+09 | 3.47e+05 |
| 3 | 7 | 1.64e+07 | 1.20e+04 | 2.26e+04 | 6 | 6 | 1.20e+09 | 3.68e+06 | 6.81e+05 |
| 3 | 9 | 1.54e+10 | 2.90e+06 | 1.33e+04 | 6 | 8 | 2.97e+12 | 1.92e+09 | 1.81e+06 |

Therefore, the refinement matrix $\mathbf{M} = \left\{m_{n,k}\right\}_{n\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^3, k\in\mathscr{I}_j^{L,1}}$ is given by

$$\mathbf{M} = \begin{pmatrix} \mathbf{C}^{-T} & \mathbf{0} \\ \mathbf{0} & \mathbf{I} \end{pmatrix} \mathbf{M}^{mon} \mathbf{C}^T. \tag{66}$$

We define the dual multiresolution spaces by

$$\tilde{S}_j := \text{span } \tilde{\Phi}_j. \tag{67}$$

**Theorem 9.** *Under the above assumptions, the following holds*
*i) The sequence* $\tilde{\mathscr{S}} = \left\{\tilde{S}_j\right\}_{j\geq j_0}$ *forms a multiresolution analysis of* $L^2([0,1])$.
*ii) The spaces* $\tilde{S}_j$ *are exact of order* $\tilde{N}$, *i.e.*

$$\Pi_{\tilde{N}-1}([0,1]) \subset \tilde{S}_j, \quad j > j_0. \tag{68}$$

*Proof* To prove $i$) we have to show the nestedness of the spaces $\tilde{S}_j$, i.e. $\tilde{S}_j \subset \tilde{S}_{j+1}$. Note that

$$\tilde{S}_j = \text{span } \tilde{\Phi}_j = \text{span } \Theta_j. \tag{69}$$

Therefore, it is sufficient to prove that any function from $\Theta_j$ can be written as a linear combination of the functions from $\Theta_{j+1}$. For the left boundary functions of the first type it is a consequence of Lemma 8. By definition (31) it holds also for the left boundary functions of the second type. Since the inner basis functions are just translated and dilated scaling function $\tilde{\phi}$, they obviously satisfy the refinement relation. Finally, right boundary scaling functions are derived by reflection from the left boundary scaling functions and therefore, they satisfy the refinement relation, too. It remains to prove that

$$\overline{\bigcup_{j\geq j_0} \tilde{S}_j} = L^2([0,1]), \tag{70}$$

where $\overline{M}$ denotes the closure of the set $M$ in $L^2([0,1])$. It is known [26] that for the spaces generated by inner functions

$$\tilde{S}_j^0 := \left\{\theta_{j,k}, k\in\mathscr{I}_j^0\right\} \tag{71}$$

we have

$$\overline{\bigcup_{j\geq j_0} \tilde{S}_j^0} = L^2([0,1]). \tag{72}$$

Hence, (70) holds independently of the choice of boundary functions.

To prove ii) we recall that the scaling function $\tilde{\phi}$ is exact of order $\tilde{N}$, i.e.

$$2^{j(r+1/2)}x^r = \sum_{k\in\mathbb{Z}} \alpha_{k,r} 2^{j/2} \tilde{\phi}\left(2^j x - k\right), \quad x\in\mathbb{R} \text{ } a.e., \text{ } r = 0,\ldots,\tilde{N}-1, \tag{73}$$

where

$$\alpha_{k,r} = \left\langle (\cdot)^k, \phi(\cdot - r)\right\rangle. \tag{74}$$

It implies that for $r = 0,\ldots,\tilde{N}-1$, $x\in\langle 0,1\rangle$, the following holds

$$\begin{aligned} 2^{j(r+1/2)}x^r|_{\langle 0,1\rangle} = & \sum_{k=-N-\tilde{N}+2}^{\tilde{N}-2} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle} + \sum_{k=\tilde{N}-1}^{2^j-N-\tilde{N}+1} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle} \\ & + \sum_{k=2^j-N-\tilde{N}+2}^{2^j+\tilde{N}-2} \alpha_{k,r} 2^{j/2}\tilde{\phi}\left(2^j x - k\right)|_{\langle 0,1\rangle}. \end{aligned}$$

By (29), (33), and (69), we immediately have

$$\Pi_{\tilde{N}-1}([0,1]) \subset \text{span}\left\{\tilde{\phi}_{j,k}, k\in\mathscr{I}_j^{L,1}\cup\mathscr{I}_j^0\cup\mathscr{I}_j^{R,1}\right\} \subset \tilde{S}_j. \tag{75}$$

## 5 Refinement Matrices

Due to the length of support of primal scaling functions, the refinement matrix $M_{j,0}$ corresponding to $\Phi$ has the following structure:

$$\mathbf{M}_{j,0} = \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{A}_j & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{76}$$

where $\mathbf{M}_L$, $\mathbf{M}_R$ are blocks $(2N-2)\times(N-1)$ and $\mathbf{A}_j$ is a $\left(2^{j+1}-N+2\right)\times\left(2^{j}-N+2\right)$ matrix given by

$$\left(\mathbf{A}_j\right)_{m,n} = \frac{1}{\sqrt{2}} h_{m-2n}, \quad 0 \le m-2n \le N. \tag{77}$$

Since the matrix $\mathbf{M}_L$ is given by

$$\begin{pmatrix} \phi_{j,-N+1} \\ \phi_{j,-N+2} \\ \vdots \\ \phi_{j,-1} \end{pmatrix} = \mathbf{M}_L^T \begin{pmatrix} \phi_{j+1,-N+1} \\ \phi_{j+1,-N+2} \\ \vdots \\ \phi_{j+1,N-1} \end{pmatrix}, \tag{78}$$

it could be computed by solving the system

$$\mathbf{P}_1 = \mathbf{M}_L^T \mathbf{P}_2, \tag{79}$$

where

$$\mathbf{P}_1 = \begin{pmatrix} \phi_{0,-N+1}(0) & \phi_{0,-N+1}(1) & \dots & \phi_{0,-N+1}(2N-3) \\ \phi_{0,-N+2}(0) & \phi_{0,-N+2}(1) & \dots & \phi_{0,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{0,-1}(0) & \phi_{0,-1}(1) & \dots & \phi_{0,-1}(2N-3) \end{pmatrix} \tag{80}$$

and

$$\mathbf{P}_2 = \begin{pmatrix} \phi_{1,-N+1}(0) & \phi_{1,-N+1}(1) & \dots & \phi_{1,-N+1}(2N-3) \\ \phi_{1,-N+2}(0) & \phi_{1,-N+2}(1) & \dots & \phi_{1,-N+2}(2N-3) \\ \vdots & & & \vdots \\ \phi_{1,N-1}(0) & \phi_{1,N-1}(1) & \dots & \phi_{1,N-1}(2N-3) \end{pmatrix}. \tag{81}$$

The solution of system (79) exists and is unique if and only if the matrix $\mathbf{P}_2$ is nonsingular. The proof of nonsingularity of $\mathbf{P}_2$ can be found [35].

To compute the refinement matrix corresponding to the dual scaling functions, we need to identify first the structure of refinement matrices $\mathbf{M}_{j,0}^{\Theta}$ corresponding to $\Theta$.

$$\mathbf{M}_{j,0}^{\Theta} = \begin{pmatrix} \mathbf{M}_L^{\Theta} & & \\ & \tilde{\mathbf{A}}_j & \\ & & \mathbf{M}_R^{\Theta} \end{pmatrix}, \tag{82}$$

where $\mathbf{M}_L^{\Theta}$ and $\mathbf{M}_R^{\Theta}$ are blocks $\left(2N+3\tilde{N}-5\right)\times\left(N+\tilde{N}-2\right)$ and $\tilde{\mathbf{A}}_j$ is a matrix of the size $\left(2^{j+1}-N-2\tilde{N}+3\right)\times\left(2^{j}-N-2\tilde{N}+3\right)$ given by

$$\left(\tilde{\mathbf{A}}_j\right)_{m,n} = \frac{1}{\sqrt{2}} \tilde{h}_{m-2n}, \quad 0 \le m-2n \le N+\tilde{N}-2. \tag{83}$$

The receipt for the computation of the refinement coefficients for the left boundary functions of the first type is the proof of Lemma 8. The refinement coefficients for the left boundary functions of the second type are given by definition (31). The matrix $\mathbf{M}_R^{\Theta}$ can be computed by the similar way.

Since we have

$$\tilde{\Phi}_j = \mathbf{Q}_j^{-T}\Theta_j = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^{\Theta}\right)^T\Theta_{j+1} = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^{\Theta}\right)^T\mathbf{Q}_{j+1}^T\tilde{\Phi}_{j+1}, \tag{84}$$

the refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to the dual scaling basis $\tilde{\Phi}_j$ is given by

$$\tilde{\mathbf{M}}_{j,0} = \mathbf{Q}_{j+1}\mathbf{M}_{j,0}^{\Theta}\mathbf{Q}_j^{-1}. \tag{85}$$

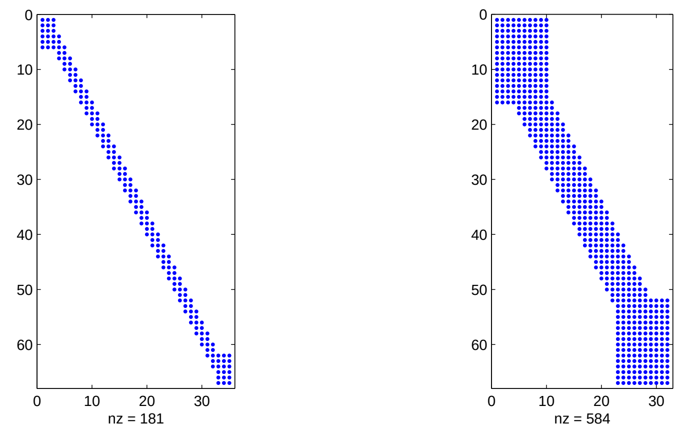



**Fig. 4** Nonzero pattern of the matrices $\mathbf{M}_{5,0}$ and $\tilde{\mathbf{M}}_{5,0}$ for $N = 4$ and $\tilde{N} = 6$, *nz* is the number of nonzero entries.

## 6 Wavelets

Our next goal is to determine the corresponding wavelet bases. This is directly connected to the task of determining an appropriate matrices $\mathbf{M}_{j,1}$ and $\tilde{\mathbf{M}}_{j,1}$. Thus, the problem has been transferred from functional analysis to linear algebra. We follow a general principle called *stable completion* which was proposed in [6].

**Definition 10.** Any $\mathbf{M}_{j,1} : l_2(J_j) \to l_2(I_{j+1})$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\left\|\mathbf{M}_j\right\|, \left\|\mathbf{M}_j^{-1}\right\| = O(1), \quad j \to \infty, \tag{86}$$

where $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$.

The idea is to determine first an initial stable completion and then to project it to the desired complement space $W_j$ determined by $\left\{\tilde{V}_j\right\}_{j\geq j_0}$. This is summarized in the following theorem [6].

**Theorem 11.** *Let $\Phi_j$ and $\tilde{\Phi}_j$ be primal and dual scaling basis, respectively. Let $\mathbf{M}_{j,0}$ and $\tilde{\mathbf{M}}_{j,0}$ be refinement matrices corresponding to these bases. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1}$. Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T\right)\check{\mathbf{M}}_{j,1} \tag{87}$$

*is also a stable completion and $\mathbf{G}_j = \mathbf{M}_j^{-1}$ has the form*

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{88}$$

*Moreover, the collections*

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1}^T \tilde{\Phi}_{j+1} \tag{89}$$

*form biorthogonal systems*

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \mathbf{0}. \tag{90}$$

We found the initial stable completion by the method from [16], [18] with some small changes. The difference is only in the dimensions of the involved matrices and in the definion of the matrix $\mathbf{F}_j$. Recall that $\mathbf{A}_j$ is the interior block in the matrix $\mathbf{M}_{j,0}$ of the form

$$\mathbf{A}_j = \frac{1}{\sqrt{2}} \begin{pmatrix} h_0 & 0 & \dots & 0 \\ h_1 & 0 & & \vdots \\ h_3 & h_0 & & \\ \vdots & \vdots & & \vdots \\ h_N & h_{N-2} & & \vdots \\ 0 & h_{N-1} & & 0 \\ 0 & h_N & & h_0 \\ \vdots & & & \vdots \\ 0 & & & h_N \end{pmatrix}, \tag{91}$$

where $h_0, \dots, h_N$ are scaling coefficients corresponding to $\phi$. By a suitable elimination we will successively reduce the upper and lower bands from $\mathbf{A}_j$ such that after $i$ steps we obtain

$$\mathbf{A}_j^{(i)} = \begin{pmatrix} 0 & 0 & & 0 \\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \\ h^{(i)}_{\lceil \frac{i}{2} \rceil} & 0 & & \\ h^{(i)}_{\lceil \frac{i}{2} \rceil + 1} & 0 & & \\ \vdots & h^{(i)}_{\lceil \frac{i}{2} \rceil} & & \\ \vdots & \vdots & & \\ h^{(i)}_{N - \lfloor \frac{i}{2} \rfloor} & & & \\ 0 & & & \\ \vdots & & & \\ & & & h^{(i)}_{N - \lfloor \frac{i}{2} \rfloor} \\ & & & 0 \\ & & & \vdots \\ 0 & & & 0 \end{pmatrix} \begin{array}{l} \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \lceil \frac{i}{2} \rceil \\ \\ \\ \\ \\ \\ \\ \\ \\ \left.\vphantom{\begin{matrix} 0 \\ \vdots \\ 0 \end{matrix}}\right\} \lfloor \frac{i}{2} \rfloor \end{array}, \quad \mathbf{A}_j^{(0)} := \mathbf{A}_j. \tag{92}$$

In [16], it was proved for B-spline scaling functions that

$$h^{(i)}_{\lceil i/2 \rceil}, \dots, h_{N - \lfloor i/2 \rfloor} \neq 0, \quad i = 1, \dots, N. \tag{93}$$

Therefore, the ellimination is always possible. The elimination matrices are of the form

$$H_j^{(2i-1)} := \operatorname{diag}\left(\mathbf{I}_{i-1}, \mathbf{U}_{2i-1}, \dots, \mathbf{U}_{2i-1}, \mathbf{I}_{N-1}\right), \tag{94}$$

$$H_j^{(2i)} := \operatorname{diag}\left(\mathbf{I}_{N-i}, \mathbf{L}_{2i}, \dots, \mathbf{L}_{2i}, \mathbf{I}_{i-1}\right), \tag{95}$$

where

$$\mathbf{U}_{i+1} := \begin{pmatrix} 1 & -\frac{h^{(i)}_{\lceil i/2 \rceil}}{h^{(i)}_{\lceil i/2 \rceil + 1}} \\ 0 & 1 \end{pmatrix}, \quad \mathbf{L}_{i+1} := \begin{pmatrix} 1 & 0 \\ -\frac{h^{(i)}_{N - \lfloor i/2 \rfloor}}{h^{(i)}_{N - \lfloor i/2 \rfloor - 1}} & 1 \end{pmatrix}. \tag{96}$$

It is easy to see that indeed

$$\mathbf{A}_j^{(i)} = \mathbf{H}_j^{(i)}\mathbf{A}_j^{(i-1)}. \tag{97}$$

After $N$ elimination steps we obtain the matrix $\mathbf{A}_j^{(N)}$ which looks as follows

$$\mathbf{A}_j^{(N)} = \mathbf{H}_j\mathbf{A}_j = \begin{pmatrix} 0 & 0 & & 0\\ \vdots & \vdots & & \vdots \\ 0 & 0 & & \\ b & 0 & & \\ 0 & 0 & & \\ 0 & b & & \\ \vdots & 0 & \ddots & \\ & & & b \\ & & & 0 \\ \vdots & & & \vdots \\ 0 & & & 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \lceil \frac{N}{2} \rceil \\ \\ \\ \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \lfloor \frac{N}{2} \rfloor \end{array}, \quad \text{where} \quad \mathbf{H}_j := \mathbf{H}_j^{(N)} \ldots \mathbf{H}_j^{(1)}, \tag{98}$$

with $b \neq 0$. We define

$$\mathbf{B}_j := \left(\mathbf{A}_j^{(N)}\right)^{-1} = \begin{pmatrix} \underbrace{0 \ldots 0}_{\lceil \frac{N}{2} \rceil} & b^{-1} & 0 & 0 & 0 \ldots & & 0 \\ 0 \ldots 0 & 0 & 0 & b^{-1} & 0 \ldots & & 0 \\ & & & & \ddots & & \\ & & & & & b^{-1} & \underbrace{0 \ldots 0}_{\lfloor \frac{N}{2} \rfloor} \end{pmatrix} \tag{99}$$

and

$$\mathbf{F}_j := \begin{pmatrix} 0 & 0 & & \\ \vdots & \vdots & & \\ 0 & 0 & & \\ 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & \ddots & \\ & & & 1 \\ & & & 0 \\ & & & \vdots \\ & & & 0 \end{pmatrix} \begin{array}{l} \left.\begin{array}{l} \\ \\ \end{array}\right\} \lceil \frac{N}{2} \rceil - 1 \\ \\ \\ \\ \\ \\ \\ \left.\begin{array}{l} \\ \\ \end{array}\right\} \lfloor \frac{N}{2} \rfloor + 1 \end{array}. \tag{100}$$

Then, we have

$$\mathbf{B}_j\mathbf{F}_j = \mathbf{0}. \tag{101}$$

After these preparations we define extended versions of the matrices $\mathbf{H}_j$, $\mathbf{A}_j$, $\mathbf{A}_j^{(N)}$, and $\mathbf{B}_j$ by

$$\hat{\mathbf{H}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{H}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \quad \hat{\mathbf{A}}_j^{(N)} := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j^{(N)} & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \tag{102}$$

$$\hat{\mathbf{A}}_j := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{A}_j & \\ & & \mathbf{I}_{N-1} \end{pmatrix}, \qquad \hat{\mathbf{B}}_j^T := \begin{pmatrix} \mathbf{I}_{N-1} & & \\ & \mathbf{B}_j^T & \\ & & \mathbf{I}_{N-1} \end{pmatrix}. \tag{103}$$

Note that $\hat{\mathbf{H}}_j$, $\hat{\mathbf{A}}_j$, $\hat{\mathbf{A}}_j^{(N)}$, and $\hat{\mathbf{B}}_j$ are all matrices of the size $(\#\mathscr{I}_{j+1}) \times (\#\mathscr{I}_j)$. Hence, the completion of $\hat{\mathbf{A}}_j^{(N)}$ has to be a $(\#\mathscr{I}_{j+1}) \times 2^j$. On the contrary to the construction in [16], we define an expanded version of $\mathbf{F}_j$ as in

[5], because it leads to a more natural formulation, when then the entries of both the refinement matrices belong to $\sqrt{2}\mathbb{Q}$. The difference is in multiplication by $\sqrt{2}$.

$$\hat{\mathbf{F}}_j := \sqrt{2}\begin{pmatrix} & \mathbf{O} & \\ \mathbf{I}_{\lceil \frac{N}{2} \rceil - 1} & & \\ & \mathbf{F}_j & \\ & & \mathbf{I}_{\lfloor \frac{N}{2} \rfloor} \\ & \mathbf{O} & \end{pmatrix} \begin{matrix} \}\, N-1 \\ \\ \\ \\ \}\, N-1 \end{matrix}. \tag{104}$$

The above findings can be summarized as follows.

**Lemma 12.** *The following relations hold:*

$$\hat{\mathbf{B}}_j \hat{\mathbf{A}}_j^{(N)} = \mathbf{I}_{\#\mathscr{I}_j}, \quad \frac{1}{2}\hat{\mathbf{F}}_j^T \hat{\mathbf{F}}_j = \mathbf{I}_{2^j} \tag{105}$$

*and*

$$\hat{\mathbf{B}}_j \hat{\mathbf{F}}_j = \mathbf{0}, \quad \hat{\mathbf{F}}_j^T \hat{\mathbf{A}}_j^{(N)} = \mathbf{0}. \tag{106}$$

The proof of this lemma is similar to the proof in [16]. Note the refinement matrix $\mathbf{M}_{j,0}$ can be factorized as

$$\mathbf{M}_{j,0} = \mathbf{P}_j \hat{\mathbf{A}}_j = \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{A}}_j^{(N)} \tag{107}$$

with

$$\mathbf{P}_j := \begin{pmatrix} \mathbf{M}_L & & \\ & \mathbf{I}_{\#\mathscr{I}_{j+1-2N}} & \\ & & \mathbf{M}_R \end{pmatrix}. \tag{108}$$

Now we are able to define the initial stable completions of the refinement matrices $\mathbf{M}_{j,0}$.

**Lemma 13.** *Under the above assumptions, the matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{P}_j \hat{\mathbf{H}}_j^{-1} \hat{\mathbf{F}}_j, \quad j \geq j_0, \tag{109}$$

*are uniformly stable completions of the matrices* $\mathbf{M}_{j,0}$. *Moreover, the inverse*

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix} \tag{110}$$

*of* $\check{\mathbf{M}}_j = \left(\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1}\right)$ *is given by*

$$\check{\mathbf{G}}_{j,0} = \hat{\mathbf{B}}_j \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}, \quad \check{\mathbf{G}}_{j,1} = \frac{1}{2}\hat{\mathbf{F}}_j^T \hat{\mathbf{H}}_j \mathbf{P}_j^{-1}. \tag{111}$$

The proof of this lemma is straightforward and similar to the proof in [16]. Then using the initial stable completion $\check{\mathbf{M}}_{j,1}$ we are already able to contruct wavelets according to the Theorem 11.

## 7 Norm equivalences

In this section, we prove norm equivalences and we show that $\Psi$ and $\tilde{\Psi}$ are Riesz bases for the space $L^2([0,1])$. Furthermore, we show that $\left\{2^{-s|\lambda|}\psi_\lambda, \lambda \in \mathscr{J}\right\}$ is a Riesz basis for Sobolev space $H^s([0,1])$ for some $s$ specified below. The proofs are based on the theory developed in [13] and [16].

Let us define

$$\gamma := \sup\left\{s : \phi \in H^s(\mathbb{R})\right\}, \quad \tilde{\gamma} := \sup\left\{s : \tilde{\phi} \in H^s(\mathbb{R})\right\}. \tag{112}$$

It is known that $\gamma = N - \frac{1}{2}$. The Sobolev exponent of smoothness $\tilde{\gamma}$ can be computed by method from [21]. The functions in $\Phi_j$ and $\Psi_j$, $j \geq j_0$, have the Sobolev regularity at least $\gamma$, because the primal scaling functions are B-splines and the primal wavelets are finite linear combinations of the primal scaling functions. Similarly, the functions in $\tilde{\Phi}_j$ and $\tilde{\Psi}_j$, $j \geq j_0$, have the Sobolev regularity at least $\tilde{\gamma}$.

**Theorem 14.** *i) The sets* $\left\{\Phi_j\right\} := \left\{\Phi_j\right\}_{j \geq j_0}$ *and* $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ *are uniformly stable, i.e.*

$$c\,\|b\|_{l_2(\mathscr{I}_j)} \leq \left\| \sum_{k \in \mathscr{I}_j} b_k \phi_{j,k} \right\| \leq C\,\|b\|_{l_2(\mathscr{I}_j)} \quad \text{for all } b = \{b_k\}_{k \in \mathscr{I}_j} \in l^2(\mathscr{I}_j),\ j \geq j_0. \tag{113}$$

*ii) For all* $j \geq j_0$, *the Jackson inequalities hold, i.e.*

$$\inf_{v_j \in S_j} \left\| v - v_j \right\| \lesssim 2^{-sj} \|v\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < N, \tag{114}$$

*and*

$$\inf_{v_j \in \tilde{S}_j} \left\| v - v_j \right\| \lesssim 2^{-sj} \|v\|_{H^s([0,1])} \quad \text{for all } v \in H^s([0,1]) \text{ and } s < \tilde{N}. \tag{115}$$

*iii) For all* $j \geq j_0$, *the Bernstein inequalities hold, i.e.*

$$\left\|v_j\right\|_{H^s([0,1])} \lesssim 2^{sj} \left\|v_j\right\| \quad \text{for all } v_j \in S_j \text{ and } s < \gamma, \tag{116}$$

*and*

$$\left\|v_j\right\|_{H^s([0,1])} \lesssim 2^{sj} \left\|v_j\right\| \quad \text{for all } v_j \in \tilde{S}_j \text{ and } s < \tilde{\gamma}. \tag{117}$$

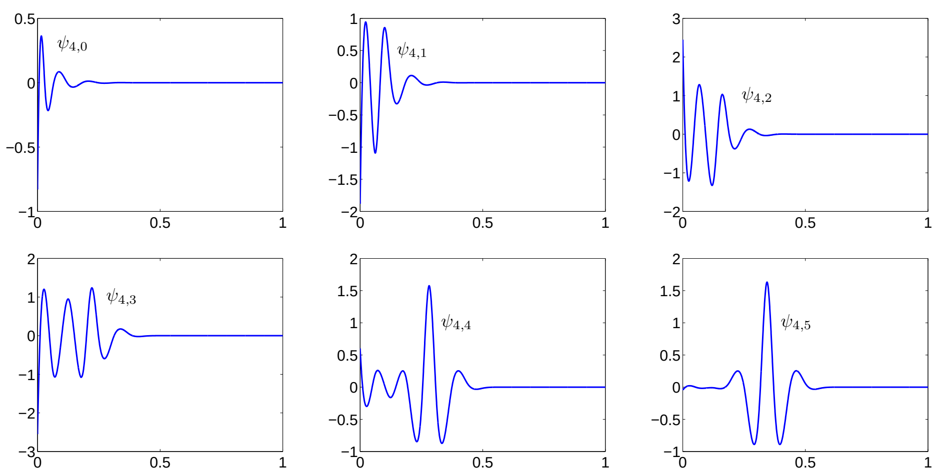



**Fig. 5** Some primal wavelets for $N = 4$ and $\tilde{N} = 6$ without boundary conditions.

*Proof* i) Due to Lemma 2.1 in [16], the collections $\{\Phi_j\} := \{\Phi_j\}_{j\geq j_0}$ and $\{\tilde{\Phi}_j\} := \{\tilde{\Phi}_j\}_{j\geq j_0}$ are uniformly stable, if $\Phi_j$ and $\tilde{\Phi}_j$ are biorthogonal,

$$\|\phi_{j,k}\| \lesssim 1, \|\tilde{\phi}_{j,k}\| \lesssim 1, \quad k \in \mathscr{I}_j,\ j \geq j_0, \tag{118}$$

and $\Phi_j$ and $\tilde{\Phi}_j$ are locally finite, i.e.

$$\#\left\{k' \in \mathscr{I}_j : \Omega_{j,k'} \cap \Omega_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathscr{I}_j,\ j \geq j_0, \tag{119}$$

and

$$\#\left\{k' \in \mathscr{I}_j : \tilde{\Omega}_{j,k'} \cap \tilde{\Omega}_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathscr{I}_j,\ j \geq j_0, \tag{120}$$

where $\Omega_{j,k} :=$ supp $\phi_{j,k}$ and $\tilde{\Omega}_{j,k} :=$ supp $\tilde{\phi}_{j,k}$.

By (39) the sets $\Phi_j$ and $\tilde{\Phi}_j$ are biorthogonal. The properties (118), (119), and (120) follow from (14), (20), and (34).

ii) By Lemma 2.1 in [16], the Jackson inequalities are the consequences of i) and the polynomial exactness (16) and (68).

iii) The Bernstein inequalities follow from i) and the regularity of basis functions, for details see [14].

The following fact follows from [13].

**Corollary 1.** *We have the norm equivalences*

$$\|v\|^2_{H^s} \sim 2^{2sj_0} \left\| \sum_{k\in\mathscr{I}_{j_0}} \left\langle v, \tilde{\phi}_{j_0,k} \right\rangle \phi_{j_0,k} \right\|^2 + \sum_{j=j_0}^{\infty} 2^{2sj} \left\| \sum_{k\in\mathscr{J}_j} \left\langle v, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k} \right\|^2, \tag{121}$$

*where $v \in H^s([0,1])$ and $s \in (-\tilde{\gamma}, \gamma)$.*

The norm equivalence for $s = 0$, Theorem 11, and Lemma 13, imply that

$$\Psi := \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j \quad \text{and} \quad \tilde{\Psi} := \tilde{\Phi}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \tilde{\Psi}_j \tag{122}$$

are biorthogonal Riesz bases of the space $L^2([0,1])$. Let us define

$$\mathbf{D} = \left(\mathbf{D}_{\lambda,\tilde{\lambda}}\right)_{\lambda,\tilde{\lambda}\in\mathscr{J}}, \quad \mathbf{D}_{\lambda,\tilde{\lambda}} := \delta_{\lambda,\tilde{\lambda}} 2^{|\lambda|}, \quad \lambda, \tilde{\lambda} \in \mathscr{J}. \tag{123}$$

The relation (121) implies that $\mathbf{D}^{-s}\Psi$ is a Riesz basis of the Sobolev space $H^s([0,1])$ for $s \in (-\tilde{\gamma}, \gamma)$.

## 8 Adaptation to Complementary Boundary Conditions

In this section, we introduce a construction of well-conditioned spline-wavelet bases on the interval satisfying complementary boundary conditions of the first order. This means that the primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the first order, whereas the dual wavelet basis preserves the full degree of polynomial exactness. This construction is based on the spline-wavelet bases constructed above. As already mentioned in Remark 7, in the linear case, i.e. $N = 2$, our bases are identical to the bases constructed in [24]. The adaptation of these bases to complementary boundary conditions can be found in [24]. Thus, we consider only the case $N \geq 3$.

Let $\Phi_j = \left\{\phi_{j,k}, k = -N+1, \ldots, 2^j - 1\right\}$ be defined as above. Note that the functions $\phi_{j,-N+1}$, $\phi_{j,2^j-1}$ are the only two functions which do not vanish at zero. Therefore, defining

$$\Phi_j^{comp} = \left\{\phi_{j,k}, k = -N+2, \ldots, 2^j - 2\right\} \tag{124}$$

we obtain primal scaling bases satisfying complementary boundary conditions of the first order.

On the dual side, we also need to omit one scaling function at each boundary, because the number of primal scaling functions must be the same as the number of dual scaling functions. Let $\Theta_j = \left\{\theta_{j,k}, k \in \mathscr{I}_j\right\}$ be the dual scaling basis on level $j$ before biorthogonalization from Section 4. There are boundary functions of two types. Recall that functions $\theta_{j,-N+1}, \ldots, \theta_{j,-N+\tilde{N}}$ are left boundary functions of the first type which are defined to preserve polynomial exactness of the order $\tilde{N}$. Functions $\theta_{j,-N+\tilde{N}+1}, \ldots, \theta_{j,\tilde{N}-2}$ are left boundary functions of the second type. The right boundary scaling functions are then derived by reflection of the left boundary functions. Since we want to preserve the full degree of polynomial exactness, we omit one function of the second type at each boundary. Thus, we define

$$\begin{aligned} \theta_{j,k}^{comp} &= \theta_{j,k-1}, \quad k = -N+2, \ldots, -N+\tilde{N}+1, \\ \theta_{j,k}^{comp} &= \theta_{j,k}, \qquad k = -N+\tilde{N}+2, \ldots, 2^j-\tilde{N}-2, \\ \theta_{j,k}^{comp} &= \theta_{j,k+1}, \quad k = 2^j-\tilde{N}-1, \ldots, 2^j-2. \end{aligned}$$

Since the set $\Theta_j^{comp} := \left\{ \theta_{j,k}^{comp} : k = -N+2, \ldots, 2^j-2 \right\}$ is not biorthogonal to $\Phi_j$, we derive a new set $\tilde{\Phi}_j^{comp}$ from $\Theta_j^{comp}$ by biorthogonalization. Let $\mathbf{Q}_j^{comp} = \left( \left\langle \phi_{j,k}, \theta_{j,l}^{comp} \right\rangle \right)_{k,l=-N+2}^{2^j-2}$, then viewing $\tilde{\Phi}_j^{comp}$ and $\Theta_j^{comp}$ as column vectors we define

$$\tilde{\Phi}_j^{comp} := \left( \mathbf{Q}_j^{comp} \right)^{-T} \Theta_j^{comp}. \tag{125}$$

Our next goal is to determine the corresponding wavelets $\Psi_j^{comp} := \left\{ \psi_{j,k}^{comp}, k = 0, \ldots, 2^j-1 \right\}$, $\tilde{\Psi}_j^{comp} := \left\{ \tilde{\psi}_{j,k}^{comp}, k = 0, \ldots, 2^j-1 \right\}$. It can be done by the method of stable completion as in Section 6.

## 9 Quantitative Properties of Constructed Bases

In this section the condition numbers of scaling bases, the single-scale wavelet bases and the multiscale wavelet bases are computed. As in [24] it can be improved by the $L^2$-normalization on the primal side. It will be shown that in the case of cubic spline wavelets bases the construction presented in this thesis yields optimal $L^2$-stability, which is not the case of constructions in [16] and [24]. The condition numbers of scaling bases and wavelet bases satisfying the complementary boundary conditions of the first order are presented as well. The other criteria for the effectiveness of wavelet bases is the condition number of the corresponding preconditioned stiffness matrix. To improve it further we apply orthogonal transformation to the scaling basis on the coarsest level and then we use a diagonal matrix for preconditioning.

It is known that Riesz bounds (2) of basis $\Phi_j$ can be computed by

$$c = \sqrt{\lambda_{min}(\mathbf{G}_j)}, \quad C = \sqrt{\lambda_{max}(\mathbf{G}_j)}, \tag{126}$$

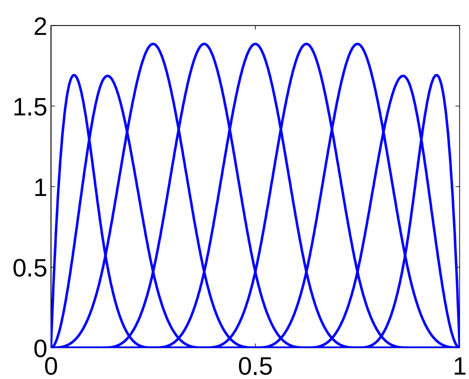



**Fig. 6** Primal scaling basis for $N = 4$ and $j = 3$ satisfying complementary boundary conditions of the first order.

where $\mathbf{G}_j$ is the Gram matrix, i.e. $\mathbf{G}_j = \left(\langle \phi_{j,k}, \phi_{j,l} \rangle\right)_{k,l \in \mathscr{I}_j}$, and $\lambda_{min}(\mathbf{G}_j)$, $\lambda_{max}(\mathbf{G}_j)$ denote the smallest and the largest eigenvalue of $\mathbf{G}_j$, respectively. The Riesz bounds of $\tilde{\Phi}_j$, $\Psi_j$ and $\tilde{\Psi}_j$ are computed in a similar way.

The condition of constructed bases is presented in Table 2. To improve it further we provide a diagonal rescaling in the following way:

$$\phi^N_{j,k} = \frac{\phi_{j,k}}{\sqrt{\langle \phi_{j,k}, \phi_{j,k} \rangle}}, \quad \tilde{\phi}^N_{j,k} = \tilde{\phi}_{j,k} * \sqrt{\langle \phi_{j,k}, \phi_{j,k} \rangle}, \quad k \in \mathscr{I}_j, \quad j \geq j_0, \tag{127}$$

$$\psi^N_{j,k} = \frac{\psi_{j,k}}{\sqrt{\langle \psi_{j,k}, \psi_{j,k} \rangle}}, \quad \tilde{\psi}^N_{j,k} = \tilde{\psi}_{j,k} * \sqrt{\langle \psi_{j,k}, \psi_{j,k} \rangle}, \quad k \in \mathscr{J}_j, \quad j \geq j_0. \tag{128}$$

Then the new primal scaling and wavelet bases are normalized with respect to the $L^2$-norm. As already mentioned in Remark 7, the resulting bases for $N = 2$ are the same as those designed in [24] and [25]. For quadratic spline-wavelet bases, i.e. $N = 3$, the condition of our bases is comparable to the condition of the bases from [24] and [25]. In [3], it was shown that for any spline wavelet basis of order $N$ on the real line, the condition is bounded below by $2^{N-1}$. This result readily carries over to the case of spline wavelet bases on the interval. Now, the constructions from [24], [25] yields wavelet bases whose Riesz bounds are nearly optimal, i.e. cond $\Psi^N_j \approx 2^{N-1}$ for $N = 2$ and $N = 3$. Unfortunately, the $L^2$-stability gets considerably worse for $N \geq 4$. As can be seen in Table 2, the column "$\Psi^N_j$", the presented construction seems to yield the optimal $L^2$-stability also for $N = 4$. Note that the case $N = 4$, $\tilde{N} = 4$ is not included in Table 2. It was shown in [9] that the corresponding scaling functions and wavelets do not belong to the space $L^2$.

In Table 3 the condition of the multiscale wavelet bases $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ is presented.

It is known that the condition number of the original basis on the real line from [9] is less than or equal to the condition number of the interval wavelet basis where the inner functions are identical to the basis functions from [9]. Therefore, we use the condition number of the wavelet bases from [9] as benchmark. In Table 4, we compare the condition number of our wavelet bases and wavelet bases from [9], [24].

The condition of single-scale bases adapted to complementary boundary condition of the first order are listed in Table 5. We improve the condition of constructed bases by $L^2$-normalization. For $N = 4$ the condition number of bases constructed in this contribution is again significantly better than the condition of bases from [24].

The other criteria for the effectiveness of wavelet bases is the condition number of the corresponding stiffness matrix. Here, let us consider the stiffness matrix for the Poisson equation:

$$\mathbf{A}_{j_0,s} = \left( \left\langle \left(\psi^{comp}_{j,k}\right)', \left(\psi^{comp}_{l,m}\right)' \right\rangle \right)_{\psi^{comp}_{j,k}, \psi^{comp}_{l,m} \in \Psi^{comp}_{j_0,s}}, \tag{129}$$

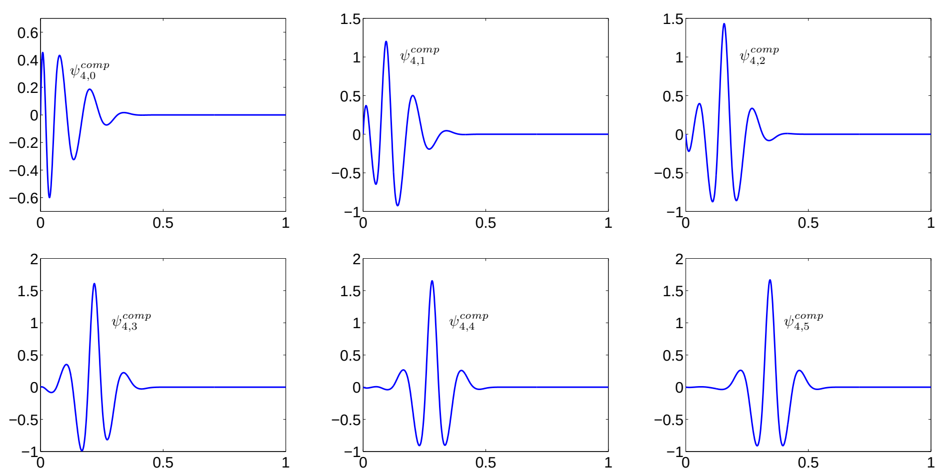



**Fig. 7** Some primal wavelets for $N = 4$ and $\tilde{N} = 6$ satisfying the complementary boundary conditions of the first order.

where $\Psi_{j_0,s}^{comp} = \Phi_{j_0}^{comp} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j^{comp}$ denotes the multiscale basis adapted to complementary boundary conditions. It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we use the diagonal matrix for preconditioning

$$\mathbf{A}_{j_0,s}^{prec} = \mathbf{D}_{j_0,s}^{-1}\mathbf{A}_{j_0,s}\mathbf{D}_{j_0,s}^{-1}, \quad \mathbf{D}_{j_0,s} = \mathrm{diag}\left(\left\langle \left(\psi_{j,k}^{comp}\right)', \left(\psi_{j,k}^{comp}\right)' \right\rangle^{1/2}\right)_{\psi_{j,k}^{comp} \in \Psi_{j_0,s}^{comp}}. \tag{130}$$

To improve further the condition number of $\mathbf{A}_{j_0,s}^{prec}$ we apply the orthogonal transformation to the scaling basis on the coarsest level as in [7] and then we use the diagonal matrix for preconditioning. We denote the obtained matrix by $\mathbf{A}_{j_0,s}^{ort}$. Condition numbers of resulting matrices are listed in Table 6.

## 10 Adaptive wavelet methods

In recent years adaptive wavelet methods have been successfully used for solving partial differential as well as integral equations, both linear and nonlinear. It has been shown that these methods converge and that they are asymptotically optimal in the sense that storage and number of floating point operations, needed to resolve the problem with desired accuracy, remain proportional to the problem size when the resolution of the discretization is refined. Thus, the computational complexity for all steps of the algorithm is controlled.

The effectiveness of adaptive wavelet methods is strongly influenced by the choice of a wavelet basis, in particular by the condition of the basis. In this section, our intention is to compare the quantitative behaviour of the adaptive wavelet method for cubic spline wavelet bases constructed in this paper and cubic spline wavelet bases from [24].

**Table 2** The condition of single-scale scaling and wavelet bases

| $N$ | $\tilde{N}$ | $j$ | $\Phi_j$ | $\Phi_j^N$ | $\tilde{\Phi}_j$ | $\tilde{\Phi}_j^N$ | $\Psi_j$ | $\Psi_j^N$ | $\tilde{\Psi}_j$ | $\tilde{\Psi}_j^N$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 10 | 2.00 | 1.73 | 2.30 | 1.97 | 2.00 | 2.00 | 2.02 | 2.00 |
| 2 | 4 | 10 | 2.00 | 1.73 | 2.09 | 1.80 | 2.00 | 2.00 | 2.04 | 2.00 |
| 2 | 6 | 10 | 2.00 | 1.73 | 2.26 | 2.03 | 2.00 | 2.00 | 2.30 | 2.26 |
| 2 | 8 | 10 | 2.00 | 1.73 | 2.90 | 2.78 | 2.34 | 2.22 | 3.14 | 3.81 |
| 3 | 3 | 10 | 3.25 | 2.76 | 7.58 | 6.37 | 4.49 | 4.00 | 7.07 | 4.27 |
| 3 | 5 | 10 | 3.25 | 2.76 | 3.93 | 3.49 | 4.63 | 4.00 | 5.55 | 4.05 |
| 3 | 7 | 10 | 3.25 | 2.76 | 3.53 | 3.11 | 4.55 | 4.00 | 5.13 | 4.01 |
| 3 | 9 | 10 | 3.25 | 2.76 | 3.75 | 3.32 | 4.44 | 4.00 | 5.51 | 4.23 |
| 4 | 6 | 10 | 5.18 | 4.42 | 10.88 | 9.07 | 14.02 | 8.00 | 24.36 | 9.23 |
| 4 | 8 | 10 | 5.18 | 4.42 | 6.69 | 5.88 | 13.96 | 8.00 | 16.98 | 8.20 |
| 4 | 10 | 10 | 5.18 | 4.42 | 5.83 | 5.16 | 13.82 | 8.00 | 15.27 | 8.00 |
| 5 | 9 | 10 | 8.32 | 7.13 | 29.87 | 25.23 | 67.74 | 27.44 | 169.76 | 68.90 |

**Table 3** The condition of the multiscale wavelet bases

| $N$ | $\tilde{N}$ | $j_0$ | $\Psi_{j_0,1}^N$ | $\Psi_{j_0,2}^N$ | $\Psi_{j_0,3}^N$ | $\Psi_{j_0,4}^N$ | $\Psi_{j_0,5}^N$ | $\tilde{\Psi}_{j_0,1}^N$ | $\tilde{\Psi}_{j_0,2}^N$ | $\tilde{\Psi}_{j_0,3}^N$ | $\tilde{\Psi}_{j_0,4}^N$ | $\tilde{\Psi}_{j_0,5}^N$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 2 | 2 | 2 | 1.98 | 2.27 | 2.52 | 2.65 | 2.76 | 2.20 | 2.42 | 2.65 | 2.78 | 2.87 |
| 2 | 4 | 3 | 2.13 | 2.25 | 2.30 | 2.33 | 2.34 | 2.15 | 2.26 | 2.31 | 2.33 | 2.35 |
| 2 | 6 | 4 | 2.47 | 2.71 | 2.84 | 2.92 | 2.99 | 2.60 | 2.78 | 2.88 | 2.94 | 3.00 |
| 2 | 8 | 4 | 3.71 | 4.77 | 5.35 | 5.68 | 5.89 | 4.44 | 5.17 | 5.57 | 5.82 | 5.98 |
| 3 | 3 | 3 | 4.92 | 6.01 | 7.15 | 7.87 | 8.50 | 7.25 | 8.54 | 9.50 | 10.08 | 10.48 |
| 3 | 5 | 4 | 4.51 | 4.82 | 5.01 | 5.10 | 5.14 | 4.63 | 4.98 | 5.11 | 5.15 | 5.16 |
| 3 | 7 | 4 | 4.19 | 4.38 | 4.44 | 4.46 | 4.48 | 4.24 | 4.39 | 4.45 | 4.48 | 4.49 |
| 3 | 9 | 5 | 4.44 | 4.55 | 4.61 | 4.64 | 4.65 | 4.48 | 4.58 | 4.62 | 4.64 | 4.66 |
| 4 | 6 | 4 | 9.55 | 10.90 | 11.88 | 12.50 | 12.90 | 10.88 | 12.90 | 13.35 | 13.48 | 13.58 |
| 4 | 8 | 5 | 8.01 | 8.31 | 8.54 | 8.68 | 8.76 | 8.23 | 8.60 | 8.73 | 8.79 | 8.81 |
| 4 | 10 | 5 | 7.89 | 8.02 | 8.09 | 8.12 | 8.13 | 7.93 | 8.05 | 8.11 | 8.13 | 8.14 |
| 5 | 9 | 5 | 30.22 | 64.60 | 75.17 | 81.03 | 84.81 | 72.34 | 83.19 | 87.93 | 90.11 | 91.27 |

**Table 4** The condition number of our multiscale wavelet bases $\Psi^N_{j_0,5}$ and $\tilde{\Psi}^N_{j_0,5}$ and multiscale wavelet bases from [9] and [24]

| $N$ | $\tilde{N}$ | $j_0$ | s | $\Psi^{CDF}_{j_0,5}$ | $\Psi^{Primbs}_{j_0,5}$ | $\Psi^N_{j_0,5}$ | $\tilde{\Psi}^{CDF}_{j_0,5}$ | $\tilde{\Psi}^{Primbs}_{j_0,5}$ | $\tilde{\Psi}^N_{j_0,5}$ |
|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 3 | 5 | 6.68 | 6.25 | 8.50 | 8.52 | 8.17 | 10.48 |
| 3 | 5 | 4 | 5 | 4.36 | 5.31 | 5.14 | 4.37 | 5.36 | 5.16 |
| 3 | 7 | 4 | 5 | 4.04 | 8.57 | 4.48 | 4.04 | 8.63 | 4.49 |
| 3 | 9 | 5 | 5 | 4.00 | 25.40 | 4.65 | 4.00 | 25.76 | 4.66 |
| 4 | 6 | 4 | 5 | 9.89 | 141.95 | 12.90 | 10.43 | 160.54 | 13.58 |
| 4 | 8 | 5 | 5 | 8.27 | 257.41 | 8.76 | 8.27 | 258.56 | 8.81 |
| 4 | 10 | 5 | 5 | 8.04 | 917.10 | 8.13 | 8.04 | 935.38 | 8.14 |
| 4 | 12 | 5 | 5 | 8.01 | 3971.65 | 8.44 | 8.01 | 3992.29 | 8.45 |

**Example 15.** We consider one-dimensional Poisson equation with homogeneous Dirichlet boundary conditions

$$-u'' = f, \quad \text{in} \quad \Omega = (0,1), \quad u(0) = u(1) = 0, \tag{131}$$

whose solution $u$ is given by

$$u(x) = 4\frac{e^{50x}-1}{e^{50}-1}\left(1-\frac{e^{50x}-1}{e^{50}-1}\right) + x(1-x), \quad x \in \Omega. \tag{132}$$

The solution exhibits steep gradient near the boundary, see Figure 8.

Let us define the diagonal matrix

$$\mathbf{D} = \operatorname{diag}\left(\left\langle \psi'_{j,k}, \psi'_{j,k}\right\rangle^{1/2}\right)_{\psi_{j,k}\in\Psi} \tag{133}$$

and operators

$$\mathbf{A} = \mathbf{D}^{-1}\left\langle \Psi', \Psi'\right\rangle \mathbf{D}^{-1}, \quad \mathbf{f} = \mathbf{D}^{-1}\left\langle f, \Psi\right\rangle. \tag{134}$$

Then the variational formulation of (131) is equivalent to

$$\mathbf{AU} = \mathbf{f} \tag{135}$$

**Table 5** The condition of scaling bases and single-scale wavelet bases satisfying complementary boundary conditions of the first order

| $N$ | $\tilde{N}$ | $j$ | $\Phi_j$ | $\Phi^N_j$ | $\tilde{\Phi}_j$ | $\tilde{\Phi}^N_j$ | $\Psi_j$ | $\Psi^N_j$ | $\tilde{\Psi}_j$ | $\tilde{\Psi}^N_j$ |
|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 10 | 2.74 | 2.74 | 4.49 | 4.34 | 4.00 | 4.00 | 4.13 | 4.00 |
| 3 | 5 | 10 | 2.74 | 2.74 | 4.94 | 4.58 | 4.00 | 4.00 | 6.68 | 6.27 |
| 3 | 7 | 10 | 2.74 | 2.74 | 8.61 | 8.33 | 4.84 | 4.27 | 12.11 | 16.05 |
| 3 | 9 | 10 | 2.74 | 2.74 | 17.94 | 17.78 | 8.16 | 6.25 | 25.17 | 46.10 |
| 4 | 6 | 10 | 4.53 | 4.31 | 7.90 | 6.83 | 9.47 | 8.00 | 16.46 | 8.00 |
| 4 | 8 | 10 | 4.53 | 4.31 | 11.16 | 10.04 | 8.46 | 8.03 | 25.40 | 15.32 |
| 4 | 10 | 10 | 4.53 | 4.31 | 17.90 | 16.97 | 8.39 | 8.42 | 37.78 | 35.93 |

**Table 6** The condition number of the stiffness matrices $\mathbf{A}^{prec}_{j,s}$, $\mathbf{A}^{ort}_{j,s}$ of the size $M \times M$

| $N$ | $\tilde{N}$ | $j$ | $s$ | $M$ | $\mathbf{A}^{prec}_{j,s}$ | $\mathbf{A}^{ort}_{j,s}$ | $N$ | $\tilde{N}$ | $j$ | $s$ | $M$ | $\mathbf{A}^{prec}_{j,s}$ | $\mathbf{A}^{ort}_{j,s}$ |
|---|---|---|---|---|---|---|---|---|---|---|---|---|---|
| 3 | 3 | 3 | 1 | 16 | 12.24 | 3.78 | 4 | 4 | 4 | 1 | 33 | 47.02 | 15.38 |
| | | | 4 | 128 | 12.82 | 5.05 | | | | 4 | 259 | 50.01 | 18.13 |
| | | | 7 | 1024 | 12.86 | 5.37 | | | | 7 | 2049 | 50.28 | 18.91 |
| 3 | 5 | 4 | 1 | 32 | 52.97 | 4.20 | 4 | 6 | 4 | 1 | 33 | 48.98 | 15.25 |
| | | | 4 | 256 | 55.09 | 8.41 | | | | 4 | 259 | 51.61 | 16.15 |
| | | | 7 | 2048 | 55.24 | 9.47 | | | | 7 | 2049 | 50.28 | 16.31 |
| 3 | 7 | 4 | 1 | 32 | 71.07 | 10.74 | 4 | 8 | 5 | 1 | 65 | 205.56 | 15.92 |
| | | | 4 | 256 | 71.90 | 33.52 | | | | 4 | 513 | 208.88 | 26.80 |
| | | | 7 | 2048 | 71.91 | 38.66 | | | | 7 | 4097 | 209.31 | 27.69 |

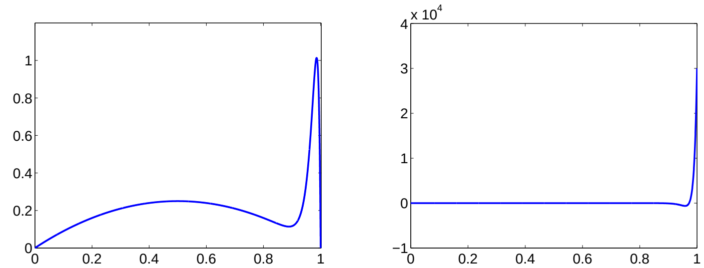



**Fig. 8** The exact solution and the right hand side of (131).

and the solution $u$ is given by $u = \mathbf{U}\mathbf{D}^{-1}\boldsymbol{\Psi}$. We solve the infinite dimensional problem (135) by the inexact damped Richardson iterations. This algorithm was originally proposed by Cohen, Dahmen and DeVore in [10]. Here, we use its modified version from [30].

Figure 9 shows a convergence history for the spline-wavelet bases designed in this contribution with $N = 4$ and $\tilde{N} = 6$ denoted by CF and the spline-wavelet bases with the same polynomial exactness from [24]. We use also the algorithm with the stiffness matrix $\mathbf{A}^{ort}$ which has lower condition number, see Table 6. Its convergence history is denoted by CFort. Note that the relative error in the energy norm for an adaptive scheme with our bases is significantly smaller even though the number of involved basis functions is about half compared with bases in [24].

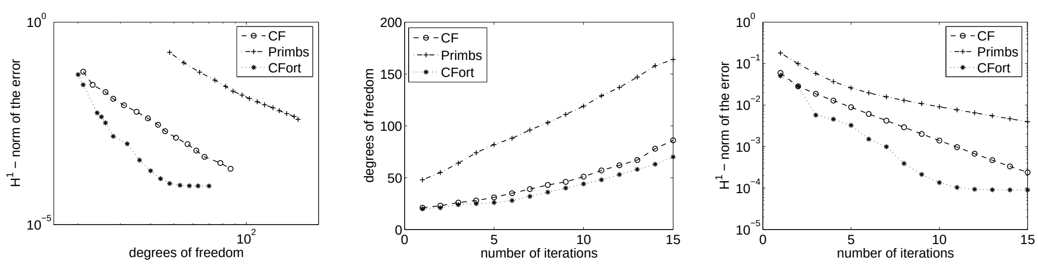



**Fig. 9** Convergence history for 1d example, comparison of our wavelet bases with and without orthogonalization and wavelet bases from [24].

**Example 16.** We consider two-dimensional Poisson equation

$$-\Delta u = f, \quad \text{in} \quad \Omega = (0,1)^2, \quad \partial\Omega = 0, \tag{136}$$

with the solution $u$ given by

$$u(x,y) = u(x)\,u(y), \quad (x,y) \in \Omega, \tag{137}$$

where $u(x)$, $u(y)$ are given by (137). We use the adaptive wavelet scheme with the cubic wavelet basis adapted to homogeneous Dirichlet boundary conditions of the first order. The convergence history for our wavelet bases with and without orthogonalization and wavelet bases from [24] is shown in Figure 10.

**Acknowledgements** The research of the first author has been supported by the Ministry of Education, Youth and Sports of the Czech Republic through the Research Centers LC06024. The second author has been supported by the research center 1M06047 of the Ministry of Education, Youth and Sports of the Czech Republic and by the grant 201/09/P641 of the Grant Agency of the Czech Republic.

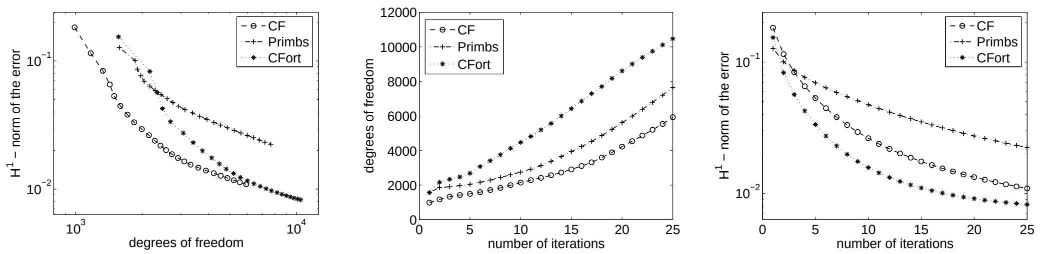



**Fig. 10** Convergence history for 2d example, comparison of our wavelet bases with and without orthogonalization and wavelet bases from [24].

## References


1. Andersson, L.; Hall, N.; Jawerth, B.; Peters, G.: *Wavelets on the Closed Subsets of the Real Line*, in: Topics in the Theory and Applications of Wavelets, L.L. Schumaker and G. Webb (eds.), Academic Press, Boston, 1994, pp. 1–64.
2. Barsch, T.: *Adaptive Multiskalenverfahren für Elliptische Partielle Differentialgleichungen - Realisierung, Umsetzung und Numerische Ergebnisse*, Ph.D. thesis, RWTH Aachen, 2001.
3. Bittner, K.: *A New View on Biorthogonal Spline Wavelets,* preprint, Universität Ulm, 2005.
4. Bittner, K.: *Biorthogonal Spline Wavelets on the Interval,* in: Wavelets and Splines, Athens, 2005, Mod. Methods Math., Nashboro Press, Brentwood, TN, 2006, pp. 93-104.
5. Burstedde, C.: *Fast Optimized Wavelet Methods for Control Problems Constrained by Elliptic PDEs,* Ph.D. thesis, Universität, Bonn, 2005.
6. Carnicer, J.M.; Dahmen, W.; Peňa, J.M.: *Local decomposition of refinable spaces,* Appl. Comp. Harm. Anal. **6**, (1999), pp. 1-52.
7. Canuto, C.; Tabacco, A.; Urban, K.: *The Wavelet Element Method, Part I: Construction and Analysis,* Appl. Comp. Harm. Anal. **6**, (1999), pp. 1-52.
8. Chui, C.K.; Quak, E.:, *Wavelets on a Bounded Interval,* in: Numerical Methods of Approximation Theory (Braess, D.; Schumaker, L.L., eds.), Birkhäuser, 1992, pp. 53-75.
9. Cohen, A.; Daubechies, I.; Feauveau, J.-C.: *Biorthogonal Bases of Compactly Supported Wavelets*. Comm. Pure and Appl. Math. **45**, (1992), pp. 485-560.
10. Cohen, A.; Daubechies, I.; Vial, P.: *Wavelets on the Interval and Fast Wavelet Transforms*, Appl. Comp. Harm. Anal. **1**, (1993), pp. 54-81.
11. Černá, D.: *Biorthogonal Wavelets,* Ph.D. thesis, Charles University, Prague, 2008.
12. Dahlke, S.; Fornasier, M.; Primbs, M.; Raasch, T.; Werner, M.: *Nonlinear and Adaptive Frame Approximation Schemes for Elliptic PDEs: Theory and Numerical Experiments,* preprint, Philipps-Universität Marburg, 2007.
13. Dahmen, W.: *Stability of Multiscale Transformations,* J. Fourier Anal. Appl. **4**, (1996), pp. 341-362.
14. Dahmen, W.: *Multiscale Analysis, Approximation, and Interpolation Spaces,* in: Approximation Theory VIII, (Chui, C.K.; Schumaker, L.L, eds.), World Scientific Publishing Co., 1995, pp. 47-88.
15. Dahmen, W.; Han, B.; Jia R.Q.; Kunoth A.: *Biorthogonal Multiwavelets on the Interval: Cubic Hermite Splines,* Constr. Approx. **16**, (2000), no. 2, pp. 221-259.
16. Dahmen, W.; Kunoth, A.; Urban, K.: *Biorthogonal Spline Wavelets on the Interval - Stability and Moment Conditions,* Appl. Comp. Harm. Anal. **6**, (1999), pp. 132-196.
17. Dahmen, W.; Kunoth, A.; Urban, K.: *Wavelets in Numerical Analysis and Their Quantitative Properties,* in: Surface fitting and multiresolution methods **2** (Le Méhauté, A.; Rabut, C.; Schumaker, L., ed.), 1997, pp. 93–130.
18. Dahmen, W.; Miccheli, C.A.: *Banded Matrices with Banded Inverses, II: Locally Finite Decomposition of Spline Spaces,* Constr. Appr. **9**, (1993), pp. 263-281.
19. Jia, R.Q.: *Stable Bases of Spline Wavelets on the Interval*, in: Wavelets and Splines, Athens 2005, Mod. Methods Math., Nashboro Press, Brentwood, TN, 2006, pp. 120-135.
20. Jia, R.Q.: *Spline Wavelets on the Interval with Homogeneous Boundary Conditions,* Adv. Comput. Math. **30**, (2009), no. 2, pp. 177–200.
21. Jia, R.Q.; Jiang, Q.T.: *Spectral Analysis of the Transition Operator and its Applications to Smoothness Analysis of Wavelets*, SIAM J. Matrix Anal. Appl. **24**, (2003), no. 4, pp. 1071–1109.
22. Jia, R.Q.; Liu, S.T.: *Wavelet Bases of Hermite Cubic Splines on the Interval,* Adv. Comput. Math. **25**, (2006), no. 1–3, pp. 23–39.
23. Pabel, R.: *Wavelet Methods for PDE Constrained Elliptic Control Problems with Dirichlet Boundary Control*, thesis, Universität Bonn, 2005.
24. Primbs, M.: *Stabile biorthogonale Spline-Waveletbasen auf dem Intervall,* dissertation, Universität Duisburg-Essen, 2006.
25. Primbs, M.: *New Stable Biorthogonal Spline Wavelets on the Interval,* preprint, Universität Duisburg-Essen, 2007.
26. Primbs, M.: *Technical Report for the Paper: 'New Stable Biorthogonal Spline Wavelets on the Interval'*, Universität Duisburg-Essen, 2007.
27. Raasch, T.: *Adaptive Wavelet and Frame Schemes for Elliptic and Parabolic Equations*, Ph.D. thesis, Marburg, 2007.
28. Schneider, A.: *Biorthogonal Cubic Hermite Spline Multiwavelets on the Interval with Complementary Boundary Conditions,* preprint, Philipps-Universität Marburg, 2007.
29. Schumaker, L.L.: *Spline Functions: Basic Theory*, Wiley-Interscience, 1981.

30. Stevenson, R.: *Adaptive Solution of Operator Equations Using Wavelet Frames*, SIAM J. Numer. Anal. **41**, (2003), pp. 1074-1100.
31. Stevenson, R.: *On the Compressibility of Operators in Wavelet Coordinates*, SIAM J. Math. Anal. **35**, (2004), no. 5, pp. 1110-1132.
32. Stevenson, R.: *Composite Wavelet Bases with Extended Stability and Cancellation Properties*, SIAM J. Numer. Anal. **45**, (2007), no. 1, pp. 133-162.
33. Grivet Talocia, S.; Tabacco, A.: *Wavelets on the Interval with Optimal Localization*, in: Math. Models Meth. Appl. Sci. **10**, 2000, pp. 441-462.
34. Turcajova, R.: *Numerical Condition of Discrete Wavelet Transforms.* SIAM J. Matrix Anal. Appl. **18**, (1997), pp. 981-999.
35. Weyrich, N.: *Spline Wavelets on an Interval*, in: Wavelets and Allied Topics, 2001, pp. 117-189.

# Cubic spline wavelets with complementary boundary conditions

Dana Černá[a], Václav Finěk[a]

[a]*Department of Mathematics and Didactics of Mathematics, Technical University in Liberec, Studentská 2, 461 17 Liberec, Czech Republic*

**Abstract**

We propose a new construction of a stable cubic spline-wavelet basis on the interval satisfying complementary boundary conditions of the second order. It means that the primal wavelet basis is adapted to homogeneous Dirichlet boundary conditions of the second order, while the dual wavelet basis preserves the full degree of polynomial exactness. We present quantitative properties of the constructed bases and we show superiority of our construction in comparison to some other known spline wavelet bases in an adaptive wavelet method for the partial differential equation with the biharmonic operator.

*Keywords:* wavelet, cubic spline, complementary boundary conditions, homogeneous Dirichlet boundary conditions, condition number
*2000 MSC:* 46B15, 65N12, 65T60

## 1. Introduction

In recent years wavelets have been successfully used for solving partial differential equations [2, 11, 12, 16, 27] as well as integral equations [22, 24, 25], both linear and nonlinear. Wavelet bases are useful in the numerical treatment of operator equations, because they are stable, enable high order-approximation, functions from Besov spaces have sparse representation in wavelet bases, condition numbers of stiffness matrices are uniformly bounded and matrices representing operators are typically sparse or quasi-sparse. The

*Email addresses:* `dana.cerna@tul.cz` (Dana Černá), `vaclav.finek@tul.cz` (Václav Finěk)

quantitative properties of wavelet methods strongly depend on the choice of a wavelet basis, in particular on its condition number. Therefore, a construction of a wavelet basis is always an important issue.

Wavelet bases on a bounded domain are usually constructed in the following way: Wavelets on the real line are adapted to the interval and then by tensor product technique to the $n$-dimensional cube. Finally by splitting the domain into overlapping or non-overlapping subdomains which are images of a unit cube under appropriate parametric mappings one can obtain a wavelet basis or a wavelet frame on a fairly general domain. Thus, the properties of the employed wavelet basis on the interval are crucial for the properties of the resulting bases or frames on a general domain.

In this paper, we propose a construction of cubic spline wavelet basis on the interval that is adapted to homogeneous Dirichlet boundary conditions of the second order on the primal side and preserves the full degree of polynomial exactness on the dual side. Such boundary conditions are called complementary boundary conditions [18]. We compare properties of wavelet bases such as the condition number of the basis and the condition number of the corresponding stiffness matrix. Finally, quantitative behaviour of adaptive wavelet method for several boundary-adapted cubic spline wavelet bases is studied.

First of all, we summarize the desired properties of a constructed basis:

- *Polymial exactness.* Since the primal basis functions are cubic B-splines, the primal multiresolution analysis has polynomial exactness of order four. The dual multiresolution analysis has polynomial exactness of order six. As a consequence, the primal wavelets have six vanishing moments.
- *Riesz basis property.* The functions form a Riesz basis of the space $L^2([0,1])$ and if scaled properly they form a Riesz basis of the space $H_0^2([0,1])$.
- *Locality.* The primal and dual basis functions are local, see definition of locality below. Then the corresponding decomposition and reconstruction algorithms are simple and fast.
- *Biorthogonality.* The primal and dual wavelet bases form a biorthogonal pair.
- *Smoothness.* The smoothness of primal and dual wavelet bases is another desired property. It ensures the validity of norm equivalences.
- *Closed form.* The primal scaling functions and wavelets are known in

the closed form. It is a desirable property for the fast computation of integrals involving primal scaling functions and wavelets.

- *Complementary boundary conditions.* Our wavelet basis satisfy complementary boundary conditions of the second order.
- *Well-conditioned bases.* Our objective is to construct a well conditioned wavelet basis.

Many constructions of cubic spline wavelet or multiwavelet bases on the interval have been proposed in recent years. In [5, 17, 26] cubic spline wavelets on the interval were constructed. In [14] cubic spline multiwavelet bases were designed and they were adapted to complementary boundary conditions of the second order in [28]. In this case dual functions are known and are local. Cubic spline wavelet bases were also constructed in [1, 9, 20, 21]. A construction of cubic spline multiwavelet basis was proposed in [19] and this basis was already used for solving differential equations in [8, 23]. However, in these cases duals are not known or are not local. Locality of duals are important in some methods and theory, let us mention construction of wavelet bases on general domain [18], adaptive wavelet methods especially for nonlinear equations, data analysis, signal and image processing. A general method of adaptation of biorthogonal wavelet bases to complementary boundary conditions was presented in [18], but this method often leads to very badly conditioned bases.

This paper is organized as follows: In Section 2 we briefly review the concept of wavelet bases. In Section 3 we propose a construction of primal and dual scaling bases. The refinement matrices are computed in Section 4 and in Section 5 primal and dual wavelets are constructed. Quantitative properties of constructed bases and other known cubic spline wavelet and multiwavelet bases are studied in Section 6. In Section 7 we compare the number of basis functions and the number of iterations needed to resolve the problem with desired accuracy for our bases and bases from [28]. A numerical example is presented for an equation with the biharmonic operator in two dimensions.

## 2. Wavelet bases

This section provides a short introduction to the concept of wavelet bases in Sobolev spaces. We consider the domain $\Omega \subset \mathbb{R}^d$ and the Sobolev space or its subspace $H \subset H^s(\Omega)$ for nonnegative integer $s$ with an inner product

$\langle \cdot, \cdot \rangle_H$, a norm $\|\cdot\|_H$ and a seminorm $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the $L^2$-inner product and the $L^2$-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale* or a *level*. Let

$$l^2(\mathcal{J}) := \left\{ \mathbf{v} : \mathcal{J} \to \mathbb{R}, \sum_{\lambda \in \mathcal{J}} |\mathbf{v}_\lambda|^2 < \infty \right\}. \tag{1}$$

A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a *wavelet basis* of $H$, if

*i)* $\Psi$ is a *Riesz basis* for $H$, i.e. the closure of the span of $\Psi$ is $H$ and there exist constants $c, C \in (0, \infty)$ such that

$$c \|\mathrm{b}\|_{l^2(\mathcal{J})} \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \leq C \|\mathrm{b}\|_{l^2(\mathcal{J})}, \quad \mathrm{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{2}$$

Constants $c_\psi := \sup\{c : c \text{ satisfies } (2)\}$, $C_\psi := \inf\{C : C \text{ satisfies } (2)\}$ are called *Riesz bounds* and *cond* $\Psi = C_\psi / c_\psi$ is called the *condition number* of $\Psi$.

*ii)* The functions are *local* in the sense that $\operatorname{diam}(\Omega_\lambda) \leq \mathrm{C}2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where $\Omega_\lambda$ is the support of $\psi_\lambda$, and at a given level $j$ the supports of only finitely many wavelets overlap at any point $x \in \Omega$.

By the Riesz representation theorem, there exists a unique family $\tilde{\Psi} = \left\{ \tilde{\psi}_\lambda, \lambda \in \tilde{\mathcal{J}} \right\} \subset H$ biorthogonal to $\Psi$, i.e.

$$\left\langle \psi_{i,k}, \tilde{\psi}_{j,l} \right\rangle_H = \delta_{i,j}\delta_{k,l}, \quad \text{for all} \quad (i,k) \in \mathcal{J}, \quad (j,l) \in \tilde{\mathcal{J}}. \tag{3}$$

This family is also a Riesz basis for $H$. The basis $\Psi$ is called a *primal* wavelet basis, while $\tilde{\Psi}$ is called a *dual* wavelet basis.

In many cases, the wavelet system $\Psi$ is constructed with the aid of a multiresolution analysis. A sequence $\mathcal{V} = \{V_j\}_{j \geq j_0}$, of closed linear subspaces $V_j \subset H$ is called a *multiresolution* or *multiscale analysis*, if

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset V_j \subset V_{j+1} \subset \ldots H \tag{4}$$

and $\cup_{j \geq j_0} V_j$ is complete in $H$.

The nestedness and the closedness of the multiresolution analysis implies the existence of the *complement spaces* $W_j$ such that $V_{j+1} = V_j \oplus W_j$.

We now assume that $V_j$ and $W_j$ are spanned by sets of basis functions

$$\Phi_j := \{\phi_{j,k}, k \in \mathcal{I}_j\}, \quad \Psi_j := \{\psi_{j,k}, k \in \mathcal{J}_j\}, \tag{5}$$

where $\mathcal{I}_j$ and $\mathcal{J}_j$ are finite or at most countable index sets. We refer to $\phi_{j,k}$ as *scaling functions* and $\psi_{j,k}$ as *wavelets*. The multiscale basis is given by $\Psi_{j_0,s} = \Phi_{j_0} \cup \bigcup_{j=j_0}^{j_0+s-1} \Psi_j$ and the wavelet basis of $H$ is obtained by $\Psi = \Phi_{j_0} \cup \bigcup_{j \geq j_0} \Psi_j$. The dual wavelet system $\tilde{\Psi}$ generates a dual multiresolution analysis $\tilde{\mathcal{V}}$ with a dual scaling basis $\tilde{\Phi}_{j_0}$.

*Polynomial exactness* of order $N \in \mathbb{N}$ for the primal scaling basis and of order $\tilde{N} \in \mathbb{N}$ for the dual scaling basis is another desired property of wavelet bases. It means that

$$\mathbb{P}_{N-1}(\Omega) \subset V_j, \quad \mathbb{P}_{\tilde{N}-1}(\Omega) \subset \tilde{V}_j, \quad j \geq j_0, \tag{6}$$

where $\mathbb{P}_m(\Omega)$ is the space of all algebraic polynomials on $\Omega$ of degree less or equal to $m$.

By Taylor theorem, the polynomial exactness of order $\tilde{N}$ on the dual side is equivalent to $\tilde{N}$ vanishing wavelet moments on the primal side, i.e.

$$\int_\Omega P(x)\, \psi_\lambda(x)\, dx = 0, \quad P \in \mathbb{P}_{\tilde{N}-1},\ \psi_\lambda \in \bigcup_{j \geq j_0} \Psi_j. \tag{7}$$

## 3. Construction of Scaling Functions

We propose a new cubic spline wavelet basis with six vanishing wavelet moments satisfying homogeneous Dirichlet boundary conditions of order two. Six vanishing wavelet moments on the primal side is equivalent to the polynomial exactness of order six on the dual side. We choose polynomial exactness of this order, because the dual scaling function of order four does not belong to $L^2(\mathbb{R})$ and the polynomial exactness of order greater than six leads to a larger support of primal wavelets which makes the computation more expensive.

The first step is the construction of primal scaling functions on the unit interval. Primal scaling basis is formed by cubic B-splines on the knots $t_k^j$ defined by

$$t_{-2}^j = t_{-1}^j := 0, \quad t_0^j := \frac{1}{2^{j+1}}, \quad t_k^j := \frac{k}{2^j}, \quad k = 1, \dots 2^j - 1, \tag{8}$$

$$t_{2^j}^j := \frac{2^{j+1}-1}{2^{j+1}}, \quad t_{2^j+1}^j = t_{2^j+2}^j := 1.$$

The corresponding cubic B-splines are defined by

$$B_k^j(x) := \left(t_{k+4}^j - t_k^j\right)\left[t_k^j, \dots, t_{k+4}^j\right]_t (t-x)_+^3, \quad x \in [0,1],$$

where $(x)_+ := \max\{0, x\}$ and $[t_1, \dots t_N]_t f$ is the $N$-th divided difference of $f$. The set $\Phi_j := \{\phi_{j,k}, k = -2, \dots, 2^j - 2\}$ of primal scaling functions is simply given by

$$\phi_{j,k} := 2^{j/2} B_k^j, \quad k = -2, \dots, 2^j - 2, \quad j \geq 0. \tag{9}$$

Thus there are $2^j - 5$ inner scaling functions and 3 boundary functions at each edge. The inner functions are translations and dilations of a function $\phi$ which corresponds to the primal scaling function constructed by Cohen, Daubechies, and Feauveau in [10]. Note that the primal scaling basis differs from the primal scaling basis constructed in [4, 5, 17, 26], because there are additional knots $\frac{1}{2^{j+1}}$ and $\frac{2^{j+1}-1}{2^{j+1}}$.

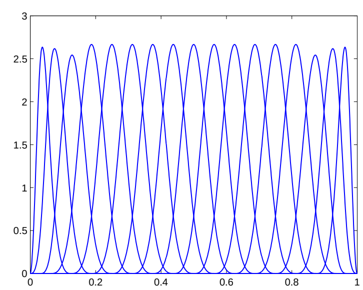



Figure 1: Primal scaling functions for the scale $j = 4$.

The desired property of a dual scaling basis $\tilde{\Phi}$ is the biorthogonality to $\Phi$ and the polynomial exactness of order six. Let $\tilde{\phi}$ be the dual scaling function which was designed by Cohen, Daubechies, and Feauveau in [10] and which is shifted so that $\tilde{\phi}$ is orthogonal to $\phi$, i.e. its support is $[-5, 9]$. It is known that there exist sequences $\{h_k\}_{k=0}^4$ and $\{\tilde{h}_k\}_{k=-5}^9$ such that the functions $\phi$ and $\tilde{\phi}$ satisfy the *refinement equations*

$$\phi(x) = \sum_{k=0}^{4} h_k \phi(2x-k), \quad \tilde{\phi}(x) = \sum_{k=-5}^{9} \tilde{h}_k \tilde{\phi}(2x-k), \quad x \in \mathbb{R}. \tag{10}$$

The parameters $h_k$ and $\tilde{h}_k$ are called *scaling coefficients*.

In the sequel, we assume that $j \geq j_0 := 4$. We define inner scaling functions as translations and dilations of $\tilde{\phi}$:

$$\theta_{j,k} = 2^{j/2}\tilde{\phi}\left(2^j \cdot -k\right), \quad k = 5, \ldots, 2^j - 9. \tag{11}$$

There will be two types of basis functions at each boundary. In the following, it will be convenient to abbreviate the boundary and inner index sets by

$$\begin{aligned} \mathcal{I}_j^{L,1} &= \{-2, \ldots, 3\}, \quad \mathcal{I}_j^{L,2} = \{4\}, \quad \mathcal{I}_j^0 = \left\{5, \ldots, 2^j - 9\right\}, \\ \mathcal{I}_j^{R,2} &= \left\{2^j - 8\right\}, \quad \mathcal{I}_j^{R,1} = \left\{2^j - 7, \ldots, 2^j - 2\right\}, \end{aligned} \tag{12}$$

and

$$\begin{aligned} \mathcal{I}_j^L &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} = \{-2, \ldots, 4\}, \\ \mathcal{I}_j^R &= \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{2^j - 8, \ldots, 2^j - 2\right\}, \\ \mathcal{I}_j &= \mathcal{I}_j^{L,1} \cup \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^0 \cup \mathcal{I}_j^{R,2} \cup \mathcal{I}_j^{R,1} = \left\{-2, \ldots, 2^j - 2\right\}. \end{aligned} \tag{13}$$

Basis functions of the first type are defined to preserve polynomial exactness and the nestedness of multiresolution spaces by the same way as in [17]:

$$\theta_{j,k}(x) = 2^{j/2} \sum_{l=-8}^{4} \langle p_{k+2}, \phi(\cdot - l) \rangle \tilde{\phi}\left(2^j x - l\right), \quad k \in \mathcal{I}_j^{L,1}, \quad x \in [0,1], \tag{14}$$

where $\{p_0, \ldots, p_5\}$ is a monomial basis of $\mathbb{P}_5([0,1])$, i.e. $p_i(x) = x^i$, $x \in [0,1]$, $i = 0, \ldots, 5$.

The definition of basis functions of the second type is a delicate task, because the low condition number and the nestedness of the multiresolution spaces have to be preserved. This means that the relation $\theta_{j,4} \in \tilde{V}_j \subset \tilde{V}_{j+1}$ should hold. Therefore we define $\theta_{j,4}$ as linear combinations of functions that are already in $\tilde{V}_{j+1}$. To obtain well-conditioned basis, we define a function $\theta_{j,4}$ which is close to $\tilde{\phi}_{j,4}^{\mathbb{R}} := 2^{j/2}\tilde{\phi}\left(2^j \cdot -4\right)$, because $\tilde{\phi}_{j,4}^{\mathbb{R}}$ is biorthogonal to the inner primal scaling functions and the condition of $\left\{\tilde{\phi}_{j,4}^{\mathbb{R}}, k \in \mathcal{I}_j^{L,2} \cup \mathcal{I}_j^0\right\}$ is close to the condition of the set of inner dual basis functions.

For this reason, we define the basis function of the second type by

$$\theta_{j,4}(x) = 2^{j/2} \sum_{l=-3}^{9} \tilde{h}_l \tilde{\phi}\left(2^{j+1}x - 8 - l\right), \quad x \in [0,1], \tag{15}$$

where $\tilde{h}_i$ are the scaling coefficients corresponding to the scaling function $\tilde{\phi}$. Then $\theta_{j,4}$ is close to $\tilde{\phi}_{j,4}^{\mathbb{R}}$ restricted to the interval $[0,1]$, because by (10) we have

$$\tilde{\phi}_{j,4}^{\mathbb{R}}(x) = 2^{j/2} \sum_{l=-5}^{9} \tilde{h}_l \tilde{\phi}\left(2^{j+1}x - 8 - l\right), \quad x \in [0,1]. \tag{16}$$

Figure 2 shows the functions $\theta_{4,4}$ and $\tilde{\phi}_{4,4}^{\mathbb{R}}$.

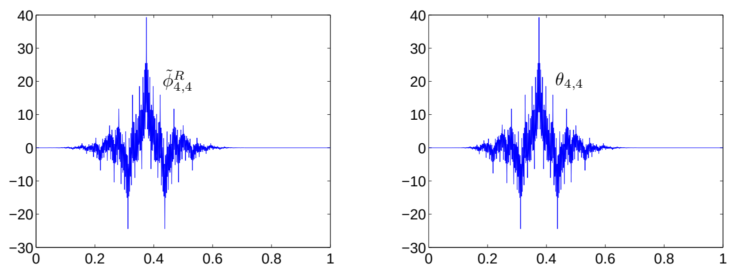



Figure 2: The functions $\tilde{\phi}_{4,4}^{\mathbb{R}}$ and $\theta_{4,4}$.

The boundary functions at the right boundary are defined to be symmetric with the left boundary functions:

$$\theta_{j,k}(x) = \theta_{j,2^j-4-k}(1-x), \quad x \in [0,1], \quad k \in \mathcal{I}_j^R. \tag{17}$$

It is easy to see that

$$\theta_{j+1,k}(x) = \sqrt{2}\,\theta_{j,k}(2x), \quad x \in [0,1], \quad k \in \mathcal{I}_j^L, \tag{18}$$

for left boundary functions and

$$\theta_{j+1,k}(1-x) = \sqrt{2}\,\theta_{j,k}(1-2x), \quad x \in [0,1], \quad k \in \mathcal{I}_j^R, \tag{19}$$

for right boundary functions.

Since the set $\Theta_j := \{\theta_{j,k}, k \in \mathcal{I}_j\}$ is not biorthogonal to $\Phi_j$, we derive a new set

$$\tilde{\Phi}_j := \left\{\tilde{\phi}_{j,k}, k \in \mathcal{I}_j\right\} \tag{20}$$

from $\Theta_j$ by biorthogonalization. Let

$$\mathbf{Q}_j = \left( \langle \phi_{j,k}, \theta_{j,l} \rangle \right)_{k,l \in \mathcal{I}_j}. \tag{21}$$

We verify numerically that $\mathbf{Q}_j$ is invertible. Viewing $\tilde{\Phi}_j$ and $\Theta_j$ as column vectors we define

$$\tilde{\Phi}_j := \mathbf{Q}_j^{-T} \Theta_j. \tag{22}$$

Then $\tilde{\Phi}_j$ is biorthogonal to $\Phi_j$, because

$$\left\langle \Phi_j, \tilde{\Phi}_j \right\rangle = \left\langle \Phi_j, \mathbf{Q}_j^{-T} \Theta_j \right\rangle = \mathbf{Q}_j \mathbf{Q}_j^{-1} = \mathbf{I}_{\# \mathcal{I}_j}, \tag{23}$$

where the symbol # denotes the cardinality of the set and $\mathbf{I}_m$ denotes the identity matrix of the size $m \times m$.

**Remark 1.** General approach of adapting wavelet bases to the unit interval was proposed in [18]. The idea is to remove certain boundary scaling functions to achieve homogeneous boundary conditions on the primal side. Then it is necessary to have the same number of basis functions on the dual side. Therefore an appropriate number of inner dual functions is used for the definition of boundary dual generators in formula (14). Applying this approach to cubic spline basis constructed in [5] and basis constructed in [26] we obtain the same resulting basis, because these constructions differs in the definition of some functions which are discarded during adaptation to complementary boundary conditions of the second order. Unfortunately, this basis has large condition number, although the starting basis in [5] is well conditioned. Its quantitative properties are presented in Section 6.

## 4. Refinement matrices

From the nestedness and the closedness of multiresolution spaces it follows that there exist *refinement matrices* $\mathbf{M}_{j,0}$ and $\mathbf{M}_{j,1}$ such that

$$\Phi_j = \mathbf{M}_{j,0}^T \Phi_{j+1}, \quad \tilde{\Phi}_j = \mathbf{M}_{j,1}^T \tilde{\Phi}_{j+1}. \tag{24}$$

Due to the length of support of primal scaling functions, the refinement matrix $\mathbf{M}_{j,0}$ has the following structure:

$$\mathbf{M}_{j,0} = \left( \begin{array}{c|c|c} \mathbf{M}_L & & \\ \hline & \mathbf{A}_j & \\ \hline & & \mathbf{M}_R \end{array} \right). \qquad (25)$$

where $\mathbf{A}_j$ is a $(2^{j+1} - 5) \times (2^j - 5)$ matrix given by

$$\begin{aligned} (\mathbf{A}_j)_{m,n} &= \frac{h_{m+1-2n}}{\sqrt{2}}, \quad n = 1, \dots, 2^j - 5,\ 0 \le m + 1 - 2n \le 4, \qquad (26) \\ &= 0, \qquad\qquad \text{otherwise,} \end{aligned}$$

where $h_m$ are primal scaling coefficients (10), and $\mathbf{M}_L$, $\mathbf{M}_R$ are given by

$$\mathbf{M}_L = \frac{1}{\sqrt{2}} \begin{pmatrix} \frac{1}{4} & 0 & 0 \\ \frac{7}{8} & \frac{1}{8} & 0 \\ \frac{1}{4} & \frac{3}{4} & 0 \\ 0 & \frac{3}{5} & \frac{2}{5} \\ 0 & \frac{3}{20} & \frac{29}{40} \\ 0 & 0 & \frac{1}{2} \\ 0 & 0 & \frac{1}{8} \end{pmatrix}, \quad \mathbf{M}_R = \mathbf{M}_L^{\updownarrow}. \qquad (27)$$

The symbol $\mathbf{M}^{\updownarrow}$ denotes a matrix that results from a matrix $\mathbf{M}$ by reversing the ordering of rows and columns. To compute the refinement matrix corresponding to the dual scaling functions, we need to identify first the structure of refinement matrices $\mathbf{M}_{j,0}^{\Theta}$ corresponding to $\Theta$:

$$\mathbf{M}_{j,0}^{\Theta} = \left( \begin{array}{c|c|c} \mathbf{M}_L^{\Theta} & & \\ \hline & \tilde{\mathbf{A}}_j & \\ \hline & & \mathbf{M}_R^{\Theta} \end{array} \right), \qquad (28)$$

where $\mathbf{M}_L^{\Theta}$ and $\mathbf{M}_R^{\Theta}$ are blocks $21 \times 7$ and $\tilde{\mathbf{A}}_j$ is a matrix of the size $(2^{j+1}-13) \times (2^j-13)$ given by

$$\begin{aligned} \left(\tilde{\mathbf{A}}_j\right)_{m,n} &= \frac{\tilde{h}_{m-2n-4}}{\sqrt{2}}, \quad n = 1, \dots 2^j - 13, \ -1 \le m-2n \le 13, \\ &= 0, \qquad\qquad\qquad \text{otherwise,} \end{aligned} \tag{29}$$

where $\tilde{h}_m$ are dual scaling coefficients (10). The refinement coefficients for the left boundary functions of the first type are computed according to the proof of Lemma 3.1 in [17]. The refinement coefficients for the left boundary functions of the second type are given by definition (15). The matrix $\mathbf{M}_R^{\Theta}$ can be computed by the similar way. Since

$$\tilde{\Phi}_j = \mathbf{Q}_j^{-T}\Theta_j = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^{\Theta}\right)^T \Theta_{j+1} = \mathbf{Q}_j^{-T}\left(\mathbf{M}_{j,0}^{\Theta}\right)^T \mathbf{Q}_{j+1}^T \tilde{\Phi}_{j+1}, \tag{30}$$

the refinement matrix $\tilde{\mathbf{M}}_{j,0}$ corresponding to the dual scaling basis $\tilde{\Phi}_j$ is given by

$$\tilde{\mathbf{M}}_{j,0} = \mathbf{Q}_{j+1}\mathbf{M}_{j,0}^{\Theta}\mathbf{Q}_j^{-1}. \tag{31}$$

## 5. Construction of wavelets

Our next goal is to determine the corresponding single-scale wavelet bases $\Psi_j$. It is directly connected to the task of determining an appropriate matrices $\mathbf{M}_{j,1}$ such that

$$\Psi_j = \mathbf{M}_{j,1}^T \Phi_{j+1}. \tag{32}$$

We follow a general principle called *stable completion* which was proposed in [3]. This approach was already used in [5, 17, 26]. In our case, however, the initial stable completion can not be found by the same way, because it leads to singular matrices.

**Definition 1.** Any $\mathbf{M}_{j,1} : l^2(\mathcal{J}_j) \to l^2(\mathcal{I}_{j+1})$ is called a *stable completion* of $\mathbf{M}_{j,0}$, if

$$\|\mathbf{M}_j\|_{l^2(\mathcal{I}_{j+1}) \to l^2(\mathcal{I}_{j+1})} = O(1), \ \left\|\mathbf{M}_j^{-1}\right\|_{l^2(\mathcal{I}_{j+1}) \to l^2(\mathcal{I}_{j+1})} = O(1), \quad j \to \infty, \tag{33}$$

where $\mathbf{M}_j := (\mathbf{M}_{j,0}, \mathbf{M}_{j,1})$.

The idea is to determine first an initial stable completion and then to project it to the desired complement space $W_j$. This is summarized in the following theorem [3].

**Theorem 2.** *Let $\Phi_j$ and $\tilde{\Phi}_j$ be a primal and a dual scaling basis, respectively. Let $\mathbf{M}_{j,0}$ and $\tilde{\mathbf{M}}_{j,0}$ be refinement matrices corresponding to these bases. Suppose that $\check{\mathbf{M}}_{j,1}$ is some stable completion of $\mathbf{M}_{j,0}$ and $\check{\mathbf{G}}_j = \check{\mathbf{M}}_j^{-1}$. Then*

$$\mathbf{M}_{j,1} := \left(\mathbf{I} - \mathbf{M}_{j,0}\tilde{\mathbf{M}}_{j,0}^T\right)\check{\mathbf{M}}_{j,1} \tag{34}$$

*is also a stable completion and $\mathbf{G}_j = \mathbf{M}_j^{-1}$ has the form*

$$\mathbf{G}_j = \begin{pmatrix} \tilde{\mathbf{M}}_{j,0}^T \\ \check{\mathbf{G}}_{j,1} \end{pmatrix}. \tag{35}$$

*Moreover, the collections*

$$\Psi_j := \mathbf{M}_{j,1}^T \Phi_{j+1}, \quad \tilde{\Psi}_j := \check{\mathbf{G}}_{j,1}\tilde{\Phi}_{j+1}, \tag{36}$$

*form biorthogonal systems*

$$\left\langle \Psi_j, \tilde{\Psi}_j \right\rangle = \mathbf{I}, \quad \left\langle \Phi_j, \tilde{\Psi}_j \right\rangle = \left\langle \Psi_j, \tilde{\Phi}_j \right\rangle = \mathbf{0}. \tag{37}$$

To find the initial stable completion we use a factorization $\mathbf{M}_{j,0} = \mathbf{H}_j \mathbf{C}_j$, where

$$\mathbf{H}_j := \begin{pmatrix} \mathbf{H}_L & & \\ & \mathbf{H}_j^I & \\ & & \mathbf{H}_R \end{pmatrix}, \tag{38}$$

$$\mathbf{H}_L := \begin{pmatrix} 0.25 & 0 & 0 & 0 & 0 \\ 0.875 & 1 & 8 & 0 & 0 \\ 0.25 & 6 & 1 & 0 & 0 \\ 0 & 4.8 & 0 & 1 & 0 \\ 0 & 1.2 & 0 & 1.8125 & 2 \\ 0 & 0 & 0 & 1.25 & 1 \\ 0 & 0 & 0 & 0.3125 & 0 \end{pmatrix}, \quad \mathbf{H}_R := \mathbf{H}_L^{\updownarrow}, \tag{39}$$

Matrix $\left(\mathbf{H}_j^I\right)$ has the size $(2^{j+1}-7) \times (2^{j+1}-9)$. Its elements are given by:

$$\begin{aligned} \left(\mathbf{H}_j^I\right)_{mn} &:= 1, \quad 1 \le n \le 2^{j+1}-9,\ n \text{ odd}, m = n+1 \\ &:= h_{2,m-n+2}^I, \quad 1 \le n \le 2^{j+1}-9,\ n \text{ even}, -1 \le m-n \le 3, \\ &:= 0, \qquad \text{otherwise}, \end{aligned} \tag{40}$$

where $h^I_{11} = h^I_{15} = 0.25$, $h^I_{12} = h^I_{14} = 1$, $h^I_{13} = 1.5$, and

$$\mathbf{C}_j := \frac{1}{\sqrt{2}} \begin{pmatrix} \mathbf{C}_L & & \\ & \mathbf{C}^I_j & \\ & & \mathbf{C}_R \end{pmatrix}, \quad \mathbf{C}_L := \begin{pmatrix} 1 & 0 & 0 \\ 0 & \frac{1}{8} & 0 \\ 0 & 0 & 0 \\ 0 & 0 & \frac{2}{5} \end{pmatrix}, \tag{41}$$

$$\mathbf{C}_R := \mathbf{C}^{\updownarrow}_L, \quad \mathbf{C}^I_j := \begin{pmatrix} 0 & 0 & & 0 \\ 0 & 0 & & \\ b & 0 & & \\ 0 & 0 & & \\ 0 & b & & \\ \vdots & 0 & \ddots & \\ & & & b \\ & & & 0 \\ 0 & & & 0 \end{pmatrix}, \quad b := \frac{7}{8}. \tag{42}$$

The factorization corresponding to inner and boundary blocks is not the same as the factorization in [15]. Therefore by our approach we obtain new inner and boundary wavelets. We define

$$\mathbf{B}_j := \sqrt{2} \begin{pmatrix} \mathbf{B}_L & & \\ & \mathbf{B}^I_j & \\ & & \mathbf{B}_R \end{pmatrix}, \mathbf{B}_L := \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 8 & 0 & 0 \\ 0 & 0 & 0 & \frac{5}{2} \end{pmatrix}, \mathbf{B}_R := \mathbf{B}^{\updownarrow}_L, \tag{43}$$

$$\mathbf{B}^I_j := \begin{pmatrix} 0 & 0 & b^{-1} & 0 & 0 & 0 & \dots & & & 0 \\ 0 & 0 & 0 & 0 & b^{-1} & 0 & \dots & & & 0 \\ & & & & & & \ddots & & & \\ & & & & & & & b^{-1} & 0 & 0 \end{pmatrix}, \tag{44}$$

and

$$\mathbf{F}_j := \begin{pmatrix} \mathbf{F}_L & & \\ & \mathbf{F}^I_j & \\ & & \mathbf{F}_R \end{pmatrix}, \tag{45}$$

$$\mathbf{F}_L := \begin{pmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{pmatrix}, \quad \mathbf{F}_R := \begin{pmatrix} 0 \\ 1 \\ 0 \end{pmatrix}, \quad \mathbf{F}_j^I := \begin{pmatrix} 1 & 0 & & \\ 0 & 0 & & \\ 0 & 1 & & \\ \vdots & 0 & \ddots & \\ & & & 1 \end{pmatrix}. \tag{46}$$

The above findings can be summarized as follows.

**Lemma 3.** *The following relations hold:*

$$\mathbf{B}_j \mathbf{C}_j = \mathbf{I}_{\#\mathcal{I}_j}, \quad \mathbf{F}_j^T \mathbf{F}_j = \mathbf{I}_{2^j}, \quad \mathbf{B}_j \mathbf{F}_j = \mathbf{0}, \quad \mathbf{F}_j^T \mathbf{C}_j = \mathbf{0}. \tag{47}$$

Now we are able to define the initial stable completions of the refinement matrices $\mathbf{M}_{j,0}$.

**Lemma 4.** *Under the above assumptions, the matrices*

$$\check{\mathbf{M}}_{j,1} := \mathbf{H}_j \mathbf{F}_j, \quad j \geq j_0, \tag{48}$$

*are uniformly stable completions of the matrices* $\mathbf{M}_{j,0}$*. Moreover, the inverse*

$$\check{\mathbf{G}}_j = \begin{pmatrix} \check{\mathbf{G}}_{j,0} \\ \check{\mathbf{G}}_{j,1} \end{pmatrix} \tag{49}$$

*of* $\check{\mathbf{M}}_j = \left(\mathbf{M}_{j,0}, \check{\mathbf{M}}_{j,1}\right)$ *is given by* $\check{\mathbf{G}}_{j,0} = \mathbf{B}_j \mathbf{H}_j^{-1}, \quad \check{\mathbf{G}}_{j,1} = \mathbf{F}_j^T \mathbf{H}_j^{-1}$.

The proof of this lemma is straightforward and similar to the proof in [17]. Then using the initial stable completion $\check{\mathbf{M}}_{j,1}$ we are already able to contruct wavelets according to the Theorem 2. Left boundary wavelets are displayed at the Figure 5.

### *5.1. Decomposition of a scaling basis on a coarse scale*

In the previous sections we assumed that the supports of the left and right boundary functions do not overlap and therefore the coarsest level was four. It might be too restrictive, especially in higher dimensions, because then there are many scaling functions. Here we decompose scaling basis $\Phi_4$ into two parts $\Phi_3$ and $\Psi_3$. It also improves the condition number of the basis. We construct wavelets on the level three to have four vanishing moments. Note that wavelets on other levels have six vanishing moments, but there the vanishing moments guaranties the smoothness of dual functions [10], and four

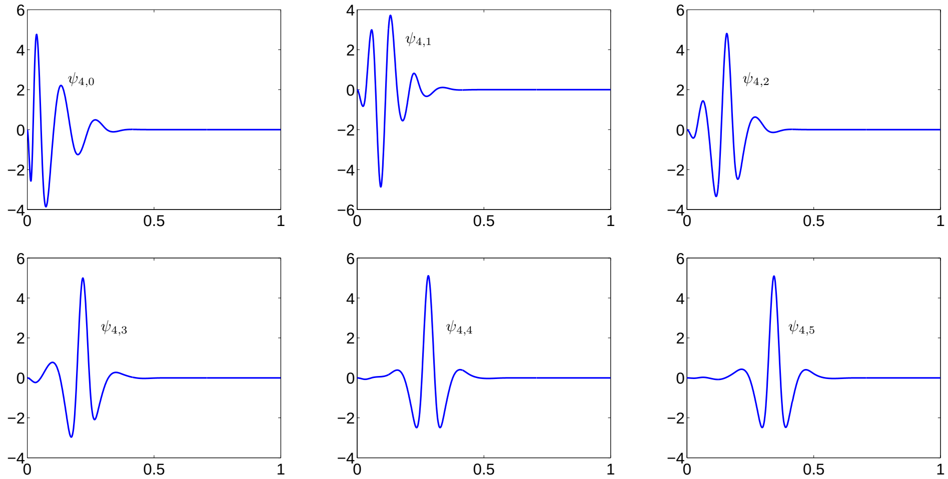



Figure 3: Left boundary wavelets for the scale $j = 4$.

vanishing moments for wavelets are sufficient in the most of the applications. Scaling functions in $\Phi_3$ are defined by (9) for $j = 3$. Functions in $\Psi_3$ are defined by

$$\psi_{3,k}(x) := \frac{(B_{t_k}^8)^{(4)}(x)}{\left\|(B_{t_k}^8)^{(4)}\right\|}, \quad k = 1, \ldots, 8, \ x \in [0,1], \tag{50}$$

where $B_{t_k}^8$ is a B-spline of order eight on the sequence of knots $t_k$ and $^{(4)}$ denotes the fourth derivative. The sequences of knots $t_k$ are given by:

$$\begin{aligned}
t_1 &= [0, 0, 1/32, 1/16, 1/8, 2/8, 3/8, 4/8, 5/8]; \qquad (51)\\
t_2 &= [0, 1/32, 1/16, 1/8, 3/16, 2/8, 3/8, 4/8, 5/8];\\
t_3 &= [1/32, 1/16, 1/8, 2/8, 5/16, 3/8, 4/8, 5/8, 6/8];\\
t_4 &= [1/16, 1/8, 2/8, 3/8, 7/16, 4/8, 5/8, 6/8, 7/8];\\
t_5 &= [1/8, 2/8, 3/8, 4/8, 9/16, 5/8, 6/8, 7/8, 15/16];\\
t_6 &= [2/8, 3/8, 4/8, 5/8, 11/16, 6/8, 7/8, 15/16, 31/32];\\
t_7 &= [3/8, 4/8, 5/8, 6/8, 13/16, 7/8, 15/16, 31/32, 1];\\
t_8 &= [3/8, 4/8, 5/8, 6/8, 7/8, 15/16, 31/32, 1, 1];
\end{aligned}$$

**Lemma 5.** *Functions from the set $\Phi_3 \cup \Psi_3$ generate the same space as functions from the set $\Phi_4$, i.e.* span $\Phi_3 \cup \Psi_3 =$ span $\Phi_4$. *Functions $\psi_{3,k}$, $k = 1, \ldots, 8$, have four vanishing wavelet moments.*

*Proof.* Since $\Phi_4$ is a basis of the space of all cubic splines on the knots $\mathbf{t}^4 = [0, 0, 1/32, 1/16, 2/16, \ldots, 15/16, 31/32, 1, 1]$. Functions in $\Phi_3$ are cubic splines on the subsets of these knots. Functions in $\Psi_3$ are also cubic splines, because they are fourth derivative of the spline of order eight, and they are defined on the subsets of knots $\mathbf{t}^4$. Therefore $\Phi_3 \cup \Psi_3 \subset \text{span}\, \Phi_4$.

Functions in $\Phi_3$ are linearly independent. Function $\psi_{3,i}$ cannot be written as linear combination of functions from $\Phi_3 \cup \Psi_3 \backslash \{\psi_{3,i}\}$, because it is a cubic spline on sequence of the knots $t_i$ containing an additional knot. Hence, $\Psi_3 \cup \Phi_3$ is a linearly independent subset of span $\Phi_4$, which proves the first assertion.

To prove that the functions $\psi_{3,k}$, $k = 1, \ldots, 8$, have four vanishing moments, we use the integration by parts. We obtain for $n = 0, \ldots, 3$:

$$\int_0^1 x^n (B^8_{t_k})^{(4)}(x)\, dx = \left[x^n \left(B^8_{t_k}\right)^{(3)}(x)\right]_0^1 - \int_0^1 n x^{n-1} \left(B^8_{t_k}\right)^{(3)}(x)\, dx. \quad (52)$$

Since $\left(B^8_{t_k}\right)^{(n)}$ is the spline of order $8-n$ on the knots of multiplicity at most two in points 0 and 1, we have

$$\left(B^8_{t_k}\right)^{(n)}(0) = \left(B^8_{t_k}\right)^{(n)}(1) = 0, \quad n = 0, \ldots 4, \quad (53)$$

and thus

$$\int_0^1 (B^8_{t_k})^{(4)}(x)\, dx = 0 \quad (54)$$

and

$$\int_0^1 x^n (B^8_{t_k})^{(4)}(x)\, dx = -\int_0^1 n x^{n-1} \left(B^8_{t_k}\right)^{(3)}(x)\, dx, \quad n = 1, \ldots, 3. \quad (55)$$

Using (53) and the integration by parts three times, we obtain:

$$\int_0^1 x^n (B^8_{t_k})^{(4)}(x)\, dx = (-1)^n\, n! \left[\left(B^8_{t_k}\right)^{(4-n)}(1) - \left(B^8_{t_k}\right)^{(4-n)}(0)\right] = 0, \quad (56)$$

for $n = 1, \ldots, 3$, which proves the assertion. □

**Remark 2.** In some constructions, the condition number of the wavelet basis is improved by orthogonalization of boundary wavelets or by the orthogonalization of scaling functions on the coarsest level. In our case, the improvement was insignificant.

*5.2. Norm equivalences*

It remains to prove that $\Psi$ and $\tilde{\Psi}$ are Riesz bases for the space $L^2([0,1])$ and that properly normalized basis $\Psi$ is a Riesz basis for Sobolev space $H^s([0,1])$ for some $s$ specified below. The proofs are based on the theory developed in [13] and [17].

For a function $f$ defined on the real line a Sobolev exponent of smoothness is defined as $\sup\{s : f \in H^s(\mathbb{R})\}$. It is known that primal scaling functions extended to the real line by zero have the Sobolev regularity at least $\gamma = \frac{5}{2}$ and that dual scaling functions extended to the real line by zero have the Sobolev regularity at least $\tilde{\gamma} = 0.344$.

**Theorem 6.** *i) The sets* $\{\Phi_j\} := \{\Phi_j\}_{j \geq j_0}$ *and* $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ *are uniformly stable, i.e.*

$$c\,\|b\|_{l_2(\mathcal{I}_j)} \leq \left\| \sum_{k \in \mathcal{I}_j} b_k \phi_{j,k} \right\| \leq C\,\|b\|_{l_2(\mathcal{I}_j)} \quad \textit{for all } b = \{b_k\}_{k \in \mathcal{I}_j} \in l^2(\mathcal{I}_j),\ j \geq j_0. \tag{57}$$

*ii) For all* $j \geq j_0$*, the Jackson inequalities hold, i.e.*

$$\inf_{v_j \in S_j} \|v - v_j\| \lesssim 2^{-sj} \|v\|_{H^s([0,1])} \quad \textit{for all } v \in H^s([0,1]) \textit{ and } s < N, \tag{58}$$

*and*

$$\inf_{v_j \in \tilde{S}_j} \|v - v_j\| \lesssim 2^{-sj} \|v\|_{H^s([0,1])} \quad \textit{for all } v \in H^s([0,1]) \textit{ and } s < \tilde{N}. \tag{59}$$

*iii) For all* $j \geq j_0$*, the Bernstein inequalities hold, i.e.*

$$\|v_j\|_{H^s([0,1])} \lesssim 2^{sj} \|v_j\| \quad \textit{for all } v_j \in S_j \textit{ and } s < \gamma, \tag{60}$$

*and*

$$\|v_j\|_{H^s([0,1])} \lesssim 2^{sj} \|v_j\| \quad \textit{for all } v_j \in \tilde{S}_j \textit{ and } s < \tilde{\gamma}. \tag{61}$$

*Proof.* i) Due to Lemma 2.1 in [17], the collections $\{\Phi_j\} := \{\Phi_j\}_{j \geq j_0}$ and $\left\{\tilde{\Phi}_j\right\} := \left\{\tilde{\Phi}_j\right\}_{j \geq j_0}$ are uniformly stable, if $\Phi_j$ and $\tilde{\Phi}_j$ are biorthogonal,

$$\|\phi_{j,k}\| \lesssim 1, \left\|\tilde{\phi}_{j,k}\right\| \lesssim 1, \quad k \in \mathcal{I}_j,\ j \geq j_0, \tag{62}$$

and $\Phi_j$ and $\tilde{\Phi}_j$ are locally finite, i.e.

$$\# \{k' \in \mathcal{I}_j : \Omega_{j,k'} \cap \Omega_{j,k} \neq \emptyset\} \lesssim 1, \quad \text{for all } k \in \mathcal{I}_j,\ j \geq j_0, \tag{63}$$

and

$$\# \left\{k' \in \mathcal{I}_j : \tilde{\Omega}_{j,k'} \cap \tilde{\Omega}_{j,k} \neq \emptyset\right\} \lesssim 1, \quad \text{for all } k \in \mathcal{I}_j,\ j \geq j_0, \tag{64}$$

where $\Omega_{j,k} := \text{supp } \phi_{j,k}$ and $\tilde{\Omega}_{j,k} := \text{supp } \tilde{\phi}_{j,k}$. By (23) the sets $\Phi_j$ and $\tilde{\Phi}_j$ are biorthogonal. The properties (62), (63), and (64) follow from (9), (11), and (18).

ii) By Lemma 2.1 in [17], the Jackson inequalities are the consequences of i) and the polynomial exactness of primal and dual multiresolution analyses.

iii) The Bernstein inequalities follow from i) and the regularity of basis functions, for details see [17].

□

The following fact follows from [13].

**Corollary 1.** *We have the norm equivalences*

$$\|v\|^2_{H^s} \sim 2^{2sj_0} \left\| \sum_{k \in \mathcal{I}_{j_0}} \left\langle v, \tilde{\phi}_{j_0,k} \right\rangle \phi_{j_0,k} \right\|^2 + \sum_{j=j_0}^{\infty} 2^{2sj} \left\| \sum_{k \in \mathcal{J}_j} \left\langle v, \tilde{\psi}_{j,k} \right\rangle \psi_{j,k} \right\|^2, \tag{65}$$

*where $v \in H^s([0,1])$ and $s \in (-\tilde{\gamma}, \gamma)$.*

The norm equivalence for $s = 0$, Theorem 2, and Lemma 4, imply that

$$\Psi := \Phi_{j_0} \cup \bigcup_{j=j_0}^{\infty} \Psi_j \quad \text{and} \quad \tilde{\Psi} := \tilde{\Phi}_{j_0} \cup \bigcup_{j=j_0}^{\infty} \tilde{\Psi}_j \tag{66}$$

are biorthogonal Riesz bases of the space $L^2([0,1])$. Let us define

$$\mathbf{D} = \left(\mathbf{D}_{\lambda,\tilde{\lambda}}\right)_{\lambda,\tilde{\lambda} \in \mathcal{J}}, \quad \mathbf{D}_{\lambda,\tilde{\lambda}} := \delta_{\lambda,\tilde{\lambda}} 2^{|\lambda|}, \quad \lambda, \tilde{\lambda} \in \mathcal{J}. \tag{67}$$

The relation (65) implies that $\mathbf{D}^{-s}\Psi$ is a Riesz basis of the Sobolev space $H^s([0,1])$ for $s \in (-\tilde{\gamma}, \gamma)$.

## 6. Quantitative properties of constructed bases

In this section, we compare quantitative properties of bases constructed in this paper, cubic spline-wavelet basis from [26] and cubic spline multiwavelet basis recently adapted to homogeneous boundary conditions in [28]. The condition of multi-scale wavelet bases is shown in Table 1. Our wavelet basis is denoted by CF, a basis from [28] is denoted by Schneider and a basis from [26] adapted to complementary boundary conditions by method from [18] is denoted by Primbs. The last basis is the same as the basis from [5] adapted to complementary boundary conditions by method from [18], see Remark 1.

Other criteria for the effectiveness of wavelet bases is the condition number of a corresponding stiffness matrix. Here, let us consider the stiffness matrix:

$$\mathbf{A}_{j_0,s} = \left(\left\langle \psi''_{j,k}, \psi''_{l,m} \right\rangle\right)_{\psi_{j,k},\psi_{l,m} \in \Psi_{j_0,s}}. \tag{68}$$

It is well-known that the condition number of $\mathbf{A}_{j_0,s}$ increases quadratically with the matrix size. To remedy this, we use a diagonal matrix for preconditioning

$$\mathbf{A}^{prec}_{j_0,s} = \mathbf{D}^{-1}_{j_0,s} \mathbf{A}_{j_0,s} \mathbf{D}^{-1}_{j_0,s}, \tag{69}$$

where

$$\mathbf{D}_{j_0,s} = \operatorname{diag}\left(\left\langle \psi''_{j,k}, \psi''_{j,k} \right\rangle^{1/2}\right)_{\psi_{j,k} \in \Psi_{j_0,s}}. \tag{70}$$

In [7] the anisotropic wavelet basis were used for solving fourth-order problems. Here, we use isotropic wavelet basis, i.e. we define multiscale wavelet basis on the unit square by

$$\Psi^{2D}_{3,s} = \Phi^{2D}_3 \cup \bigcup_{j=3}^{s} \Psi^{2D}_j, \tag{71}$$

Table 1: The condition numbers of wavelet bases and stiffness matrices, $j_0 = 3$ for CF and Schneider, $j_0 = 4$ for Primbs.

| | $\Psi_{j_0,j}$ | | | $\mathbf{A}^{prec}_{j_0,j}$ | | |
|---|---|---|---|---|---|---|
| j | CF | Schneider | Primbs | CF | Schneider | Primbs |
| 1 | 8.3 | 1.9 | 14.9 | 64.8 | 472.0 | 1111.0 |
| 3 | 12.5 | 2.4 | 45.9 | 66.5 | 569.5 | 1116.9 |
| 5 | 15.3 | 2.6 | 69.8 | 66.6 | 640.8 | 1117.0 |
| 7 | 18.0 | 2.7 | 85.8 | 66.7 | 693.0 | 1117.0 |

where

$$\Phi_3^{2D} = \Phi_3 \otimes \Phi_3, \quad \Psi_j^{2D} = \Phi_j \otimes \Psi_j \cup \Psi_j \otimes \Phi_j \cup \Psi_j \otimes \Psi_j. \tag{72}$$

The symbol $\otimes$ denotes the tensor product. The preconditioned stiffness matrix ${}^{2D}\mathbf{A}_{j_0,s}^{prec}$ for the biharmonic equation defined on the unit square is similar to the one dimensional case. Condition numbers of the stiffness matrices are listed in Table 1 and Table 2. The condition number of the stiffness matrix corresponding to wavelet basis by Primbs exceeds $10^4$ already for number of levels $j = 3$. Wavelet basis from [17] adapted to complementary boundary conditions by method from [18] is very badly conditioned, its quantitative properties can be found in [28].

## 7. Numerical example

Now, we compare the quantitative behaviour of the adaptive wavelet method with our bases and bases from [28]. Both bases are formed by cubic splines and have local duals. We consider the equation

$$\Delta^2 u = f \quad \text{in} \quad \Omega, \quad u = \frac{\partial u}{\partial n} = 0 \quad \text{on} \quad \partial\Omega, \tag{73}$$

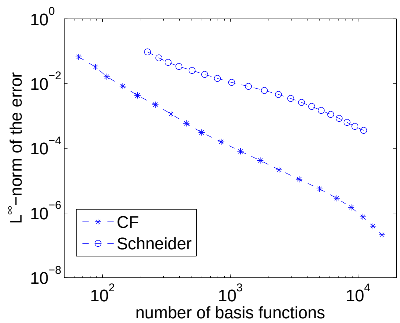
for $\Omega = (0,1)^2$, where the solution $u$ is given by

$$u(x,y) = v(x)\, v(y), \quad v(x) := x^2 \left(1 - \frac{e^{10x}}{e^{10}}\right)^2. \tag{74}$$

Note that the solution exhibits a sharp gradient near the point $[1,1]$. We solve the problem by the method designed in [12] with the approximate

Table 2: The condition of numbers of stiffness matrices of the size $N \times N$ for $j$ levels.

| j | N | CF | N | Schneider |
|---|---:|---:|---:|---:|
| 1 | 289 | 128.05 | 900 | 484.35 |
| 2 | 1089 | 141.28 | 3844 | 583.41 |
| 3 | 4225 | 212.01 | 15876 | 626.91 |
| 4 | 16641 | 257.56 | 64516 | 653.45 |
| 5 | 66049 | 281.21 | 260100 | 673.19 |
| 6 | 263169 | 297.23 | 1044484 | 689.43 |
| 7 | 1050625 | 306.12 | 4186116 | 703.42 |



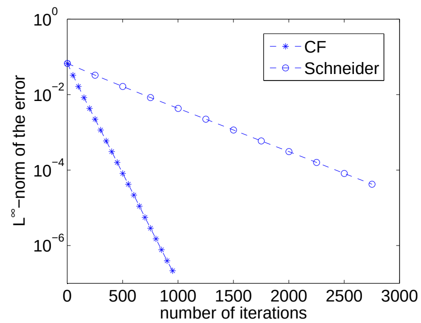



Figure 4: The convergence history for adaptive wavelet scheme with various wavelet bases.

multiplication of the stiffness matrix with a vector proposed in [6]. We use wavelets up to the scale $|\lambda| \leq 10$. The convergence history is shown in Figure 4. In our experiments, the convergence rate, i.e. the slope of the curve, is similar for both bases. However, they significantly differ in the number of basis functions and number of iterations needed to resolve the problem with desired accuracy. The number of basis functions was about $10^4$ for an error in $L^\infty$-norm about $10^{-7}$. The number of all basis functions for full grid, i.e. basis functions on the level ten or less, is about $10^6$, therefore by using an adaptive method the significant compression was achieved. It can seem that the number of iterations is quite large, but one could take into account that in the beginning the iterations were done for much smaller vector and the size of the vector increases successively. The algorithm is asymptotically optimal, i.e. the computational time depends linearly on the number of basis functions, see [12].

**Acknowledgments:** The authors have been supported by the project ESF "Constitution and improvement of a team for demanding technical computations on parallel computers at TU Liberec" No. CZ.1.07/2.3.00/09.0155.

## References


[1] K. Bittner, Biorthogonal spline wavelets on the interval, in: Wavelets and Splines, Athens, 2005, Mod. Methods Math., Nashboro Press, Brentwood, TN, 2006, pp. 93-104.

[2] N.M. Bujurke, C.S. Salimath, R.B. Kudenatti, S.C. Shiralashetti, A fast

wavelet-multigrid method to solve elliptic partial differential equations, Appl. Math. Comput. 185 (2007) 667-680.

[3] J.M. Carnicer, W. Dahmen, J.M. Peña, Local decomposition of refinable spaces, Appl. Comp. Harm. Anal. 3 (1996), 127-153.

[4] D. Černá, V. Finěk, Optimized construction of biorthogonal spline-wavelets, in: T.E. Simos et al. (Eds.), ICNAAM 2008, AIP Conference Proceedings 1048, American Institute of Physics, New York, 2008, pp. 134-137.

[5] D. Černá, V. Finěk, Construction of optimally conditioned cubic spline wavelets on the interval, Adv. Comput. Math. 34 (2011), 519-552.

[6] D. Černá, V. Finěk, Approximate multiplication in adaptive wavelet methods, submitted.

[7] D. Černá, V. Finěk, Adaptive wavelet method for fourth-order elliptic problems, in: T.E. Simos et al. (Eds.), ICNAAM 2011, AIP Conference Proceedings 1389, American Institute of Physics, New York, 2011, pp. 1606-1609.

[8] X.F. Chen, J.W. Xiang, Solving diffusion equation using wavelet method, Appl. Math. Comput. 217 (2011) 6426-6432.

[9] C.K. Chui, E. Quak, Wavelets on a bounded interval, in: Numerical Methods of Approximation Theory (Braess, D.; Schumaker, L.L., eds.), Birkhäuser, 1992, pp. 53-75.

[10] A. Cohen, I. Daubechies, J. C. Feauveau, Biorthogonal bases of compactly supported wavelets, Comm. Pure and Appl. Math. 45 (1995) 485-560.

[11] A. Cohen, W. Dahmen, R. DeVore, Adaptive wavelet schemes for elliptic operator equations - convergence rates, Math. Comput. 70 (2001) 27-75.

[12] A. Cohen, W. Dahmen, R. DeVore, Adaptive wavelet methods II - beyond the elliptic case, Found. Math. 2 (2002) 203-245.

[13] W. Dahmen, Stability of multiscale transformations, J. Fourier Anal. Appl. 4 (1996) 341-362.

[14] W. Dahmen, B. Han, R.Q. Jia, A. Kunoth, Biorthogonal multiwavelets on the interval: cubic Hermite splines, Constr. Approx. 16 (2000) 221-259.

[15] W. Dahmen, C.A. Miccheli, Banded matrices with banded inverses, II: locally finite decomposition of spline spaces, Constr. Appr. 9 (1993) 263-281.

[16] S. Dahlke, W. Dahmen, K. Urban, Adaptive wavelet methods for saddle point problems - optimal convergence rates, SIAM J. Numer. Anal. 40 (2002) 1230-1262.

[17] W. Dahmen, A. Kunoth, K. Urban, Biorthogonal spline wavelets on the interval - stability and moment conditions, Appl. Comp. Harm. Anal. 6 (1999) 132-196.

[18] W. Dahmen, R. Schneider, Wavelets with complementary boundary conditions - function spaces on the cube, Results Math. 34 (1998) 255-293.

[19] R.Q. Jia, S.T. Liu, Wavelet bases of Hermite cubic splines on the interval, Adv. Comput. Math 25 (2006) 23-39.

[20] R.Q. Jia,, Spline wavelets on the interval with homogeneous boundary conditions, Adv. Comput. Math 30 (2009) 177-200.

[21] R.Q. Jia, W. Zhao, Riesz bases of wavelets and applications to numerical solutions of elliptic equations, Math. Comput. 80 (2011) 1525-1556.

[22] A. Khademi, K. Maleknejad, Filter matrix based on interpolation wavelets for solving Fredholm integral equations, Commun. Nonlinear Sci. Numer. Simul. 16 (2011) 4197-4207.

[23] S.T. Liu, Y. Xu, Graded Galerkin methodds for high-order convection diffusion problem, Numer. Methods Partial Differential Eq. 25 (2009) 1261-1282.

[24] K. Maleknejad, M.N. Sahlan, The method of moments for solution of second kind Fredholm integral equations based on B-spline wavelets, Int. J. Comput. Math. 87 (2010) 1602-1616.

[25] K. Maleknejad, M. Yousefi, Numerical solution of the integral equation of the second kind by using wavelet bases of Hermite cubic splines, Appl. Math. Comput. 183 (2006) 134-141.

[26] M. Primbs, New stable biorthogonal spline-wavelets on the interval, Result. Math. 57 (2010) 121-162.

[27] K. Sandeep, S. Gaurb, D. Duttac, H.S. Kushwahac, Wavelet based schemes for linear advectiondispersion equation, Appl. Math. Comput. 218 (2011) 3786-3798.

[28] A. Schneider, Biorthogonal cubic Hermite spline multiwavelets on the interval with complementary boundary conditions, Result. Math. 53 (2009) 407-416.

International Journal of Wavelets, Multiresolution and Information Processing


# WAVELET BASIS OF CUBIC SPLINES ON THE HYPERCUBE SATISFYING HOMOGENEOUS BOUNDARY CONDITIONS

DANA ČERNÁ * and VÁCLAV FINĚK

*Department of Mathematics and Didactics of Mathematics, Technical University of Liberec*
*Studentská 2, 461 17 Liberec, Czech Republic*
*dana.cerna@tul.cz*
*vaclav.finek@tul.cz*

Received (Day Month Year)
Revised (Day Month Year)
Accepted (Day Month Year)
Published (Day Month Year)

In the paper, we propose a construction of a new cubic spline-wavelet basis on the hypercube satisfying homogeneous Dirichlet boundary conditions. Wavelets have two vanishing moments. Stiffness matrices arising from discretization of elliptic problems using a constructed wavelet basis have uniformly bounded condition numbers and we show that these condition numbers are small. We present quantitative properties of the constructed basis and we provide a numerical example to show an efficiency of Galerkin method using constructed basis.

Keywords: Construction; Wavelet; Cubic spline; Homogeneous Dirichlet boundary conditions; Condition number; Elliptic problem; Galerkin method; Conjugate gradient method.

AMS Subject Classification: 46B15, 65N12, 65T60

## 1. Introduction

In this paper, we propose a construction of a new cubic spline wavelet basis on the hypercube that is well-conditioned, adapted to homogeneous Dirichlet boundary conditions and the wavelets have two vanishing moments. The wavelet basis of the space $H_0^1(\Omega)$, where $\Omega = (0,1)^d$ and $d \in \mathbb{N}$, is then obtained by a tensor product and a proper normalization.

First of all, we summarize the desired properties of a constructed basis:

- *Riesz basis property.* We construct Riesz basis of the space $L^2(\Omega)$ that, when normalized with respect to $H^1$-seminorm, is also a Riesz basis of the space $H_0^1(\Omega)$.

*Corresponding author

- *Polymial exactness.* Since the primal basis functions are cubic B-splines, the primal multiresolution analysis has polynomial exactness of order four. It means that all polynomials of degree less than four belong to the span of scaling functions at the given level.
- *Vanishing moments.* The wavelets have two vanishing moments.
- *Locality.* The primal basis functions are local in the sense of Definition 1.1 below.
- *Smoothness.* Primal basis functions belong to $C^2(\Omega)$ and dual basis functions belong to $C(\Omega)$, where $C(\Omega)$ is the space of continuous functions on domain $\Omega$ and $C^n(\Omega)$ is the space of functions on domain $\Omega$ that have continuous derivatives up to order $n \in \mathbb{N}$.
- *Closed form.* The primal scaling functions and wavelets are known in the closed form.
- *Homogeneous Dirichlet boundary conditions.* Constructed wavelet basis satisfies homogeneous Dirichlet boundary conditions.
- *Well-conditioned bases.* Our objective is to construct a wavelet basis that is well conditioned with respect to the $L_2$-norm and is well conditioned with respect to the $H^1$-seminorm, when normalized appropriately.

We denote the Sobolev space or its subspace by $H \subset H^s(\Omega)$ for nonnegative integer $s$ and the corresponding inner product by $\langle \cdot, \cdot \rangle_H$, a norm by $\|\cdot\|_H$ and a seminorm by $|\cdot|_H$. In case $s = 0$ we consider the space $L^2(\Omega)$ and we denote by $\langle \cdot, \cdot \rangle$ and $\|\cdot\|$ the $L^2$-inner product and the $L^2$-norm, respectively. Let $\mathcal{J}$ be some index set and let each index $\lambda \in \mathcal{J}$ take the form $\lambda = (j, k)$, where $|\lambda| := j \in \mathbb{Z}$ is a *scale.* Let

$$\|\mathbf{v}\|_2 := \sqrt{\sum_{\lambda \in \mathcal{J}} v_\lambda^2}, \quad \text{for} \quad \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, v_\lambda \in \mathbb{R}, \tag{1.1}$$

and

$$l^2(\mathcal{J}) := \left\{ \mathbf{v} : \mathbf{v} = \{v_\lambda\}_{\lambda \in \mathcal{J}}, v_\lambda \in \mathbb{R}, \|\mathbf{v}\|_2 < \infty \right\}. \tag{1.2}$$

Our aim is to construct a wavelet basis in the sence of the following definition.

**Definition 1.1.** *A family $\Psi := \{\psi_\lambda, \lambda \in \mathcal{J}\}$ is called a (primal) wavelet basis of $H$, if*

*i)* *$\Psi$ is a Riesz basis for $H$, i.e. the closure of the span of $\Psi$ is $H$ and there exist constants $c, C \in (0, \infty)$ such that*

$$c\|\mathbf{b}\|_2 \le \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda \psi_\lambda \right\|_H \le C \|\mathbf{b}\|_2, \quad \textit{for all} \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}). \tag{1.3}$$

*Constants $c_\Psi := sup\{c : c$ satisfies (1.3)$\}$, $C_\Psi := inf\{C : C$ satisfies (1.3)$\}$ are called Riesz bounds and cond $\Psi = C_\Psi / c_\Psi$ is called the condition number of $\Psi$.*

*ii)* *The functions are local in the sense that* $diam\,(\Omega_\lambda) \leq C2^{-|\lambda|}$ *for all* $\lambda \in \mathcal{J}$*, where* $\Omega_\lambda$ *is the support of* $\psi_\lambda$*, and at a given level* $j$ *the supports of only finitely many wavelets overlap at any point* $x \in \Omega$*.*

For the two countable sets of functions $\Gamma, \Omega \subset H$ , the symbol $\langle \Gamma, \Omega \rangle_H$ denotes the matrix

$$\langle \Gamma, \Omega \rangle_H := \{\langle \gamma, \omega \rangle_H\}_{\gamma \in \Gamma, \omega \in \Omega} . \tag{1.4}$$

**Remark 1.1.** It is known that the constants $c_\Psi$ and $C_\Psi$ from Definition 1.1 satisfy:

$$c_\Psi = \sqrt{\lambda_{min}\left(\langle \Psi, \Psi \rangle_H\right)}, \quad C_\Psi = \sqrt{\lambda_{max}\left(\langle \Psi, \Psi \rangle_H\right)}, \tag{1.5}$$

where $\lambda_{min}\left(\langle \Psi, \Psi \rangle_H\right)$ and $\lambda_{max}\left(\langle \Psi, \Psi \rangle_H\right)$ are the smallest and the largest eigenvalues of the matrix $\langle \Psi, \Psi \rangle_H$, respectively.

Many constructions of spline wavelet or multiwavelet bases on the interval have been proposed in recent years.[3,4,15,18,19,21] In Ref. 1, 2, 11, 17 cubic spline wavelets on the interval were constructed. In these cases dual functions are known and are local. Spline wavelet or multiwavelet bases where duals are not local are also known.[5,12–15] The advantage of our construction in comparison with cubic spline biorthogonal wavelets with local duals[1,2,11,17] is that the support of wavelets is shorter, condition numbers of the corressponding stiffness matrices are smaller and the advantage is also a simple construction.

## 2. Construction of scaling functions

A primal scaling basis is the same as a scaling basis in Ref. 1, 17. It is generated from functions $\phi$, $\phi_{b1}$ and $\phi_{b2}$. Let $\phi$ be a cubic B-spline defined on knots $\{0, 1, 2, 3, 4\}$. It can be written explicitly as:

$$\phi(x) = \begin{cases} \frac{x^3}{6}, & x \in [0,1], \\ -\frac{x^3}{2} + 2x^2 - 2x + \frac{2}{3}, & x \in [1,2], \\ \frac{x^3}{2} - 4x^2 + 10x - \frac{22}{3}, & x \in [2,3], \\ \frac{(4-x)^3}{6}, & x \in [3,4], \\ 0, & \text{otherwise}. \end{cases} \tag{2.1}$$

Then $\phi$ satisfies *a scaling equation*[1,17] :

$$\phi\left(x\right) = \frac{\phi\left(2x\right)}{8} + \frac{\phi\left(2x-1\right)}{2} + \frac{3\phi\left(2x-2\right)}{4} + \frac{\phi\left(2x-3\right)}{2} + \frac{\phi\left(2x-4\right)}{8}. \tag{2.2}$$

Let $\phi_{b1}$ be a cubic B-spline defined on knots $\{0, 0, 0, 1, 2\}$ and $\phi_{b2}$ be a cubic B-spline defined on knots $\{0, 0, 1, 2, 3\}$, i.e.,

$$\phi_{b1}(x) = \begin{cases} \frac{7x^3}{4} - \frac{9x^2}{2} + 3x, & x \in [0,1], \\ \frac{(2-x)^3}{4}, & x \in [1,2], \\ 0, & \text{otherwise}, \end{cases} \tag{2.3}$$

and

$$\phi_{b2}(x) = \begin{cases} -\frac{11x^3}{12} + \frac{3x^2}{2}, & x \in [0,1], \\ \frac{7x^3}{12} - 3x^2 + \frac{9x}{2} - \frac{3}{2}, & x \in [1,2], \\ \frac{(3-x)^3}{6}, & x \in [2,3], \\ 0, & \text{otherwise}. \end{cases} \tag{2.4}$$

Then $\phi_{b1}$ and $\phi_{b2}$ satisfy scaling equations:[1,17]

$$\begin{aligned} \phi_{b1}(x) &= \frac{\phi_{b1}(2x)}{2} + \frac{3\phi_{b2}(2x)}{4} + \frac{3\phi(2x)}{16}, \\ \phi_{b2}(x) &= \frac{\phi_{b2}(2x)}{4} + \frac{11\phi(2x)}{16} + \frac{\phi(2x-1)}{2} + \frac{\phi(2x-2)}{8}. \end{aligned} \tag{2.5}$$

For $j \in \mathbb{N}$, $j \geq 3$ and $x \in [0,1]$ we set

$$\phi_{j,k}(x) = 2^{j/2}\phi(2^j x - k),\ k = 3, \ldots 2^j - 1, \tag{2.6}$$

$$\begin{aligned} \phi_{j,1}(x) &= 2^{j/2}\phi_{b1}(2^j x), \quad \phi_{j,2^j+1}(x) = 2^{j/2}\phi_{b1}(2^j(1-x)), \\ \phi_{j,2}(x) &= 2^{j/2}\phi_{b2}(2^j x), \quad \phi_{j,2^j}(x) = 2^{j/2}\phi_{b2}(2^j(1-x)). \end{aligned} \tag{2.7}$$

Furthermore, we define

$$\Phi_j = \left\{\phi_{j,k}/\left\|\phi_{j,k}\right\|, k = 1, \ldots, 2^j + 1\right\} \quad \text{and} \quad V_j = \operatorname{span} \Phi_j. \tag{2.8}$$

The sets $\Phi_j$ are uniform Riesz bases of the space $V_j$. It means that the sets $\Phi_j$ are Riesz bases of the space $V_j$ with Riesz bounds independent on $j$. The proof can be found in Ref. 1. The graphs of the functions $\phi_{j,k}$ on the coarsest level $j = 3$ are displayed in Figure 1.

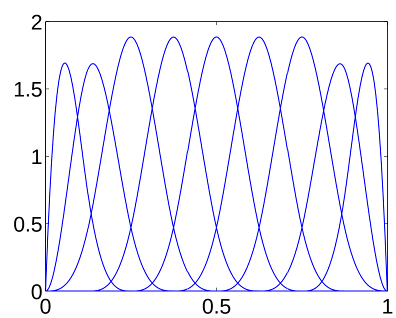



Fig. 1. Functions $\phi_{3,k}$, $k = 1, \ldots, 9$.

## 3. Construction of wavelets

In some applications such as adaptive wavelet methods,[6,7] vanishing moments of wavelets are needed. In our case, we construct wavelets with two vanishing moments, i.e.

$$\int_{-\infty}^{\infty} x^k \psi(x) dx = 0, \quad k = 0, 1. \tag{3.1}$$

We set $\tilde{V}_j$ as the space of continuous piecewise linear function:

$$\tilde{V}_j = C\left(0,1\right) \cap \prod_{k=0}^{2^j-1} P_1\left(\frac{k}{2^j}, \frac{k+1}{2^j}\right), \tag{3.2}$$

where $P_1\left(a,b\right)$ is the space of all algebraic polynomials on $(a,b)$ of degree less or equal to 1. Clearly, with this choice the dimension of $\tilde{V}_j$ is $2^j+1$ that is the same as the dimension of $V_j$. We construct wavelets $\psi_{j,k}$, $k=1,\ldots,2^j$, such that $\psi_{j,k} \in V_{j+1}$ and

$$\left\langle \psi_{j,k}, \tilde{\phi} \right\rangle = 0 \tag{3.3}$$

for all functions $\tilde{\phi} \in \tilde{V}_j$, because then (3.1) will be satisfied.

Since we want $\psi_{j,k} \in V_{j+1}$, we define a generator wavelet $\psi$ as

$$\psi(x) = \sum_{k=0}^{6} g_k \phi(2x-k), \tag{3.4}$$

and

$$[g_0, \ldots, g_6] = \left[\frac{-1}{184}, \frac{7}{46}, \frac{-119}{184}, 1, \frac{-119}{184}, \frac{7}{46}, \frac{-1}{184}\right]. \tag{3.5}$$

The coefficients $g_k$ are computed such that $\langle \psi, \omega \rangle = 0$ for all functions $\omega$ that are continuous and are linear on intervals $[k, k+1]$, $k \in \mathbb{Z}$. Then for

$$\psi_{j,k}(x) = 2^{j/2}\psi(2^j x - k + 2),\ k = 3, ..., 2^j-2,\ j \in \mathbb{N},\ j \geq 3, \tag{3.6}$$

the condition (3.3) is satisfied and the functions $\psi$ and $\psi_{j,k}$ have two vanishing wavelet moments. The support of the wavelet $\psi$ is $[0,5]$. The graph of $\psi$ is shown in Figure 2.

We define boundary wavelets $\psi_{b1}$ and $\psi_{b2}$ by:

$$\psi_{b1}(x) = g_0^{b1}\phi_{b1}(2x) + g_1^{b1}\phi_{b2}(2x) + \sum_{k=2}^{4} g_k^{b1}\phi(2x-k+2), \tag{3.7}$$

$$\psi_{b2}(x) = g_0^{b2}\phi_{b1}(2x) + g_1^{b2}\phi_{b2}(2x) + \sum_{k=2}^{6} g_k^{b2}\phi(2x-k+2),$$

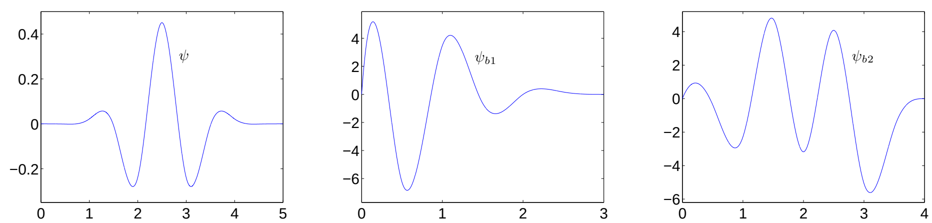



Fig. 2. Wavelets $\psi$, $\psi_{b1}$ and $\psi_{b2}$.

where

$$\left[g_0^{b1},\dots,g_4^{b1}\right]=\left[\frac{939}{70},\frac{-393}{20},\frac{6233}{560},-4,1\right], \tag{3.8}$$

$$\left[g_0^{b2},\dots,g_6^{b2}\right]=\left[\frac{2770661}{1828560},\frac{256057}{457140},\frac{-493633}{76992},\frac{20761777}{1828560},\frac{-76369591}{7314240},7,-3\right].$$

Then $\operatorname{supp}\psi_{b1} = [0,3]$, $\operatorname{supp}\psi_{b2} = [0,4]$ and both boundary wavelets have two vanishing moments.

For $j \in \mathbb{N}$, $j \geq 3$ and $x \in [0,1]$ we define

$$\begin{aligned}\psi_{j,1}(x) &= 2^{j/2}\psi_{b1}(2^j x), \quad \psi_{j,2^j}(x) = 2^{j/2}\psi_{b1}(2^j(1-x)),\\ \psi_{j,2}(x) &= 2^{j/2}\psi_{b2}(2^j x), \quad \psi_{j,2^j-1}(x) = 2^{j/2}\psi_{b2}(2^j(1-x)).\end{aligned} \tag{3.9}$$

and

$$\Psi_j = \left\{\psi_{j,k}/\left\|\psi_{j,k}\right\|, k=1,\dots,2^j\right\}, \quad W_j = \operatorname{span}\Psi_j. \tag{3.10}$$

We denote

$$\Psi^s = \Phi_3 \cup \bigcup_{j=3}^{2+s}\Psi_j \quad \text{and} \quad \Psi = \Phi_3 \cup \bigcup_{j=3}^{\infty}\Psi_j. \tag{3.11}$$

In the following, we prove that $\Psi$ is Riesz basis of the space $L_2(0,1)$. The set $\Psi^s$ is a finite dimensional approximation of $\Psi$.

**Theorem 3.1.** *The sets $\Psi_j$, $j \geq 3$, are uniform Riesz bases of $W_j$.*

**Proof.** We computed the matrix

$$\mathbf{F}_j := \langle \Psi_j,\ \Psi_j \rangle \tag{3.12}$$

using (3.4) and (3.7). For example, for $j = 3$ we obtained

$$\mathbf{F}_3 = \begin{pmatrix} 1.000 & 0.128 & 0.103 & 0.003 & 0 & 0 & 0 & 0 \\ 0.128 & 1.000 & 0.432 & -0.145 & -0.014 & 0 & 0 & 0 \\ 0.103 & 0.432 & 1.000 & -0.029 & -0.077 & 0.001 & 0 & 0 \\ 0.003 & -0.145 & -0.029 & 1.000 & -0.029 & -0.077 & -0.014 & 0 \\ 0 & -0.014 & -0.077 & -0.029 & 1.000 & -0.029 & -0.145 & 0.003 \\ 0 & 0 & 0.001 & -0.077 & -0.029 & 1.000 & 0.432 & 0.103 \\ 0 & 0 & 0 & -0.014 & -0.145 & 0.432 & 1.000 & 0.128 \\ 0 & 0 & 0 & 0 & 0.003 & 0.103 & 0.128 & 1.000 \end{pmatrix}, \tag{3.13}$$

where the numbers are rounded to three decimal places. The matrix $\mathbf{F}_j$ for $j \geq 3$ has the similar structure. The first two rows and columns and the last two rows and columns corresponds to boundary wavelets and for $k, l = 3, \ldots 2^j - 2$:

$$(\mathbf{F}_j)_{k,l} = \begin{cases} 1, & k = l, \\ -0.029, & |k-l| = 1, \\ -0.077, & |k-l| = 2, \\ -0.001, & |k-l| = 3, \\ 0, & \text{otherwise.} \end{cases} \tag{3.14}$$

It is easy to see that $\mathbf{F}_j$ is banded and diagonally dominant. Estimates for the smallest eigenvalue $\lambda^j_{min}$ and the largest eigenvalue $\lambda^j_{max}$ of the matrix $\mathbf{F}_j$ can be computed using the Gershgorin circle theorem:

$$\lambda^j_{min} \geq \min \left( \left| F^j_{ii} \right| - \sum_{k=1}^{n} \left| F^j_{ik} \right| \right) > 0.2, \tag{3.15}$$

$$\lambda^j_{max} \leq \max \left( \left| F^j_{ii} \right| + \sum_{k=1}^{n} \left| F^j_{ik} \right| \right) < 1.8, \tag{3.16}$$

$$\tag{3.17}$$

where $F^j_{ik}$ are the entries of the matrix $\mathbf{F}_j$. With the help of Remark 1.1 the assertion is proven. □

The proof that $\Psi$ is a Riesz basis is based on the following theorem.[8,12]

**Theorem 3.2.** *Let $J \in \mathbb{N}$ and let $V_j$ and $\tilde{V}_j$, $j \geq J$, be subspaces of $L_2(0,1)$ such that*

$$V_j \subset V_{j+1}, \quad \tilde{V}_j \subset \tilde{V}_{j+1}, \quad \text{dim} V_j = \text{dim} \tilde{V}_j < \infty, \quad j \geq J. \tag{3.18}$$

*Let $\Phi_j$ be uniform Riesz bases of $V_j$, $\tilde{\Phi}_j$ be uniform Riesz bases of $\tilde{V}_j$, $\Psi_j$ be uniform Riesz bases of $\tilde{V}_j^{\perp} \cap V_{j+1}$, where $\tilde{V}_j^{\perp}$ is the orthogonal complement of $\tilde{V}_j$ with respect to the $L^2$-inner product, and let*

$$\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} = \Phi_J \cup \bigcup_{j=J}^{\infty} \Psi_j. \tag{3.19}$$

*Furthermore, let the matrix*

$$\mathbf{G}_j := \left\langle \Phi_j, \tilde{\Phi}_j \right\rangle \tag{3.20}$$

*be invertible and the spectral norm of* $\mathbf{G}_j^{-1}$ *is bounded independently on* $j$. *In addition, for some positive constants* $C$, $\gamma$ *and* $d$, $\gamma < d$, *let*

$$\inf_{v_j \in V_j} \|v - v_j\| \leq C 2^{-jd} \|v\|_{H^d(0,1)}, \; v \in H_0^d(0,1), \tag{3.21}$$

*and for* $0 \leq s < \gamma$ *let*

$$\|v_j\|_{H^d(0,1)} \leq C 2^{js} \|v_j\|, \; v_j \in V_j, \tag{3.22}$$

*and let similar estimates (3.21) and (3.22) hold for* $\tilde{\gamma}$ *and* $\tilde{d}$ *on the dual side. Then there exist constants* $k$ *and* $K$, $0 < k \leq K < \infty$, *such that*

$$k \|\mathbf{b}\|_2 \leq \left\| \sum_{\lambda \in \mathcal{J}} b_\lambda 2^{-|\lambda| s} \psi_\lambda \right\|_{H^s(0,1)} \leq K \|\mathbf{b}\|_2, \quad \mathbf{b} := \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l^2(\mathcal{J}) \tag{3.23}$$

*holds for* $s \in (-\tilde{\gamma}, \gamma)$.

**Theorem 3.3.** *The set* $\Psi$ *is a wavelet basis of the space* $L_2(0,1)$.

**Proof.** We consider the set

$$\bar{\Phi}_j = \left\{ \phi_{j,k}, k = 1, \ldots, 2^j \right\} \tag{3.24}$$

that is a Riesz basis of the space $V_j$. Recall that $\tilde{V}_j$ is defined by (3.2). Let

$$\tilde{\phi}(x) = \begin{cases} x+1, \, x \in [-1,0], \\ 1-x, \, x \in [0,1], \\ \quad 0, \quad \text{otherwise}, \end{cases} \tag{3.25}$$

and for $x \in [0,1]$ we define

$$\tilde{\phi}_{j,k}(x) = 2^{j/2} \tilde{\phi}\left(2^j x - k\right), k = 1, \ldots, 2^j - 1, \tag{3.26}$$

$$\tilde{\phi}_{j,k}(x) = 2^{(j+1)/2} \tilde{\phi}\left(2^j x - k\right), k = 0, 2^j. \tag{3.27}$$

Then

$$\tilde{\Phi}_j = \left\{ \tilde{\phi}_{j,k}, k = 0, \ldots, 2^j \right\} \tag{3.28}$$

are uniform Riesz basis of the space $\tilde{V}_j$.[1]

The matrix $\mathbf{G}_j = \left\langle \bar{\Phi}_j, \tilde{\Phi}_j \right\rangle$ has the structure

$$\mathbf{G}_j = \begin{pmatrix} \frac{17}{40} & \frac{11}{40} & \frac{11}{80} & 0 & 0 & 0 & 0 & & 0 \\ \frac{19}{120} & \frac{9}{20} & \frac{17}{80} & \frac{11}{120} & 0 & 0 & 0 & & 0 \\ \frac{1}{60} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 & 0 & & 0 \\ 0 & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & & & 0 \\ 0 & 0 & \ddots & \ddots & \ddots & \ddots & \ddots & & \vdots \\ \vdots & & & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{120} & 0 \\ 0 & & & & \frac{1}{120} & \frac{13}{60} & \frac{11}{20} & \frac{13}{60} & \frac{1}{60} \\ 0 & & & & & \frac{11}{120} & \frac{17}{80} & \frac{9}{20} & \frac{19}{120} \\ 0 & & & & & & \frac{11}{80} & \frac{11}{40} & \frac{17}{40} \end{pmatrix}. \tag{3.29}$$

It is easy to verify that the matrix $\mathbf{G}_j$ is banded and strictly diagonally dominant. Therefore, it is invertible and the spectral norm of $\mathbf{G}_j^{-1}$ is bounded independently on $j$. It is known[8] that when $\gamma$ is the Sobolev exponent of smoothness of the basis functions and $d$ is the polynomial exactness of $V_j$ than (3.21) and (3.22) are satisfied. In our case, the Sobolev exponent of smoothness is $\gamma = 3.5$ and the polynomial exactness of $V_j$ is $d = 4$. On the dual side, $\tilde{\gamma} = 1.5$ and $\tilde{d} = 2$. Therefore, due to Theorem 3.2, the norm equivalence (3.23) is satisfied for $s \in (-1.5, 3.5)$. Since we proved that (3.23) holds for $s = 0$, the set $\Psi$ is indeed a wavelet basis of the space $L_2(0,1)$. □

It remains to prove that when the wavelet basis $\Psi$ is normalized in the $H^1$-seminorm, then it is a wavelet basis of the space $H_0^1(0,1)$. We denote

$$\mathcal{I}_3 := \{0, 1, \ldots, 8\} \quad \text{and} \quad \mathcal{J}_j := \left\{1, \ldots, 2^j\right\}. \tag{3.30}$$

**Theorem 3.4.** *The set*

$$\left\{\phi_{3,k} / \left|\phi_{3,k}\right|_{H_0^1(0,1)}, k \in \mathcal{I}_3\right\} \cup \left\{\psi_{j,k} / \left|\psi_{j,k}\right|_{H_0^1(0,1)}, j \geq 3, k \in \mathcal{J}_j\right\} \tag{3.31}$$

*is a wavelet basis of the space* $H_0^1(0,1)$.

**Proof.** We follow the Proof of Theorem 2 in Ref. 3. From the proof of Theorem 3.3, we know that the relation (3.23) holds for $s = 1$. Therefore the set

$$\left\{2^{-3}\phi_{3,k}, k \in \mathcal{I}_3\right\} \cup \left\{2^{-j}\psi_{j,k}, j \geq 3, k \in \mathcal{J}_j\right\} \tag{3.32}$$

is a wavelet basis of the space $H_0^1(0,1)$. From (2.6), (3.6) and (3.9) there exist nonzero constants $C_1$ and $C_2$ such that

$$C_1 2^j \leq \left|\psi_{j,k}\right|_{H_0^1(\Omega)} \leq C_2 2^j, \quad \text{for} \quad j \geq 3, \quad k \in \mathcal{J}_j, \tag{3.33}$$

and

$$C_1 2^3 \leq \left|\phi_{3,k}\right|_{H_0^1(\Omega)} \leq C_2 2^3, \quad \text{for } k \in \mathcal{I}_3. \tag{3.34}$$

Let $\hat{\mathbf{b}} = \{\hat{a}_{3,k},\ k \in \mathcal{I}_3\} \cup \left\{\hat{b}_{j,k}, j \geq 3,\ k \in \mathcal{J}_j\right\}$ be such that

$$\left\|\hat{\mathbf{b}}\right\|_2^2 = \sum_{k \in \mathcal{I}_3} \hat{a}_{3,k}^2 + \sum_{k \in \mathcal{J}_j, j \geq 3} \hat{b}_{j,k}^2 < \infty. \tag{3.35}$$

We define

$$a_{3,k} = \frac{2^3 \hat{a}_{3,k}}{|\phi_{3,k}|_{H_0^1(0,1)}},\ k \in \mathcal{I}_3, \quad b_{j,k} = \frac{2^j \hat{b}_{j,k}}{|\psi_{j,k}|_{H_0^1(0,1)}},\ j \geq 3,\ k \in \mathcal{J}_j, \tag{3.36}$$

and $\mathbf{b} = \{a_{3,k},\ k \in \mathcal{I}_3\} \cup \{b_{j,k}, j \geq 3,\ k \in \mathcal{J}_j\}$. Then

$$\|\mathbf{b}\|_2 \leq \frac{\left\|\hat{\mathbf{b}}\right\|_2}{C_1} < \infty. \tag{3.37}$$

Since (3.32) is a Riesz basis of $H_0^1(0,1)$ there exist constants $C_3$ and $C_4$ such that

$$C_3 \|\mathbf{b}\|_2 \leq \left\| \sum_{k \in \mathcal{I}_3} a_{3,k} 2^{-3} \phi_{3,k} + \sum_{k \in \mathcal{J}_j, j \geq 3} b_{j,k} 2^{-j} \psi_{j,k} \right\|_{H_0^1(0,1)} \leq C_4 \|\mathbf{b}\|_2 . \tag{3.38}$$

Therefore

$$\frac{C_4}{C_1} \left\|\hat{\mathbf{b}}\right\|_2 \geq C_4 \|\mathbf{b}\|_2 \geq \left\| \sum_{k \in \mathcal{I}_3} a_{3,k} 2^{-3} \phi_{3,k} + \sum_{k \in \mathcal{J}_j, j \geq 3} b_{j,k} 2^{-j} \psi_{j,k} \right\|_{H_0^1(0,1)} \tag{3.39}$$

$$= \left\| \sum_{k \in \mathcal{I}_3} \frac{\hat{a}_{3,k}}{|\phi_{3,k}|_{H_0^1(0,1)}} \phi_{3,k} + \sum_{k \in \mathcal{J}_j, j \geq 3} \frac{\hat{b}_{3,k}}{|\psi_{j,k}|_{H_0^1(0,1)}} \psi_{j,k} \right\|_{H_0^1(0,1)}$$

and similarly

$$\frac{C_3}{C_2} \left\|\hat{\mathbf{b}}\right\|_2 \leq \left\| \sum_{k \in \mathcal{I}_3} \frac{\hat{a}_{3,k}}{|\phi_{3,k}|_{H_0^1(0,1)}} \phi_{3,k} + \sum_{k \in \mathcal{J}_j, j \geq 3} \frac{\hat{b}_{3,k}}{|\psi_{j,k}|_{H_0^1(0,1)}} \psi_{j,k} \right\|_{H_0^1(0,1)} . \tag{3.40}$$

□

It is known[1,16] that an orthogonalization of the scaling functions on the coarsest level can lead to improved quantitative properties of the resulting wavelet basis. Therefore, we define the set

$$\Phi_3^{ort} = \left\{\phi_{3,k}^{ort},\ k \in \mathcal{I}_3\right\} \tag{3.41}$$

by

$$\Phi_3^{ort} := \mathbf{K}^{-1} \Phi_3, \quad \mathbf{K} = \langle \Phi_3, \Phi_3 \rangle . \tag{3.42}$$

Then the set of scaling functions $\Phi_3^{ort}$ is orthonormal and

$$\Psi^{ort} := \Phi_3^{ort} \cup \bigcup_{j=3}^{\infty} \Psi_j \tag{3.43}$$

is a wavelet basis of the space $L^2(0,1)$ and its appropriate rescaling is a wavelet basis of the space $H_0^1(0,1)$.

## 4. Multivariate wavelets

We present two well-known constructions of multivariate wavelet bases on the unit hypercube.[22] They are both based on tensorizing univariate wavelet bases and preserve Riesz basis property, locality of wavelets, vanishing moments and polynomial exactness.

### 4.1. *Anisotropic construction*

For notational simplicity, we denote

$$\psi_{2,k} := \phi^{ort}_{3,k}, \quad k \in \mathcal{J}_2 := \mathcal{I}_3 \tag{4.1}$$

and

$$\mathcal{J} := \{(j,k)\,,\ j \geq 2,\ k \in \mathcal{J}_j\}\,. \tag{4.2}$$

Then we can write

$$\Psi^{ort} = \{\psi_{j,k},\ j \geq 2,\ k \in \mathcal{J}_j\} = \{\psi_\lambda,\ \lambda \in \mathcal{J}\}\,. \tag{4.3}$$

Recall that for $\lambda = (j,k)$ we denote $|\lambda| = j$. We use $u \otimes v$ to denote the tensor product of functions $u$ and $v$, i.e. $(u \otimes v)(x_1, x_2) = u(x_1)\, v(x_2)$. We define multivariate basis functions as:

$$\psi_{\boldsymbol{\lambda}} = \otimes_{i=1}^{d} \psi_{\lambda_i}, \quad \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d) \in \mathbf{J}, \quad \mathbf{J} = \mathcal{J}^d = \mathcal{J} \otimes \ldots \otimes \mathcal{J}. \tag{4.4}$$

Since $\Psi^{ort}$ is a Riesz basis of $L_2(0,1)$ and $\Psi^{ort}$ normalized with respect to $H^1$-seminorm is a Riesz basis of $H^1_0(0,1)$, the set

$$\Psi^{ani} := \{\psi_{\boldsymbol{\lambda}}, \boldsymbol{\lambda} \in \mathbf{J}\} \tag{4.5}$$

is a Riesz basis of $L_2(\Omega)$, $\Omega = (0,1)^d$, and its normalization

$$\left\{ \frac{\psi_{\boldsymbol{\lambda}}}{|\psi_{\boldsymbol{\lambda}}|_{H^1\left((0,1)^d\right)}}, \boldsymbol{\lambda} \in \mathbf{J} \right\} \tag{4.6}$$

is a Riesz basis of $H^1_0(\Omega)$. The set

$$\Psi^{ani}_s := \{\psi_{\boldsymbol{\lambda}}, \boldsymbol{\lambda} = (\lambda_1, \ldots, \lambda_d)\,,\ |\lambda_i| < 2 + s\} \tag{4.7}$$

is a finite-dimensional approximation of $\Psi^{ani}$.

### 4.2. *Isotropic construction*

We define for $j \geq 3$ and $\mathbf{k} = (k_1, \ldots k_d)$ multivariate scaling functions:

$$\phi_{j,\mathbf{k}} := \otimes_{i=1}^{d} \phi_{j,k_i}, \tag{4.8}$$

and

$$\Phi^{iso}_j := \{\phi_{j,\mathbf{k}},\ \mathbf{k} = (k_1, \ldots k_d)\,,\ k_i \in \mathcal{I}_j,\ i = 1, \ldots, d\}\,. \tag{4.9}$$

For $e \in \{0,1\}$ we define

$$\psi_{j,k,e} = \begin{cases} \phi_{j,k}, \ e = 0, \\ \psi_{j,k}, \ e = 1. \end{cases} \tag{4.10}$$

We denote the index set:

$$\mathcal{J}_{j,e} = \begin{cases} \mathcal{I}_j, \ e = 0, \\ \mathcal{J}_j, \ e = 1. \end{cases} \tag{4.11}$$

For $\mathbf{k} = (k_1, \ldots k_d)$ and $\mathbf{e} = (e_1, \ldots, e_d)$ we define multivariate functions

$$\psi_{j,\mathbf{k},\mathbf{e}} = \otimes_{i=1}^{d} \psi_{j,k_i,e_i} \tag{4.12}$$

and the set of wavelets on the level $j$ as

$$\Psi_j^{iso} = \left\{ \psi_{j,\mathbf{k},\mathbf{e}}, \ k_i \in \mathcal{J}_{j,e_i}, \ \mathbf{e} \in E \right\}, \quad \text{where } E = \{0,1\}^d \setminus \{\mathbf{0}\}. \tag{4.13}$$

It is known that then the set

$$\Psi_{iso} = \Phi_3^{iso} \cup \bigcup_{j=3}^{\infty} \Psi_j^{iso} \tag{4.14}$$

is a wavelet basis of $L_2(\Omega)$ and its normalization with respect to the $H^1(\Omega)$-seminorm is a Riesz basis of $H_0^1(\Omega)$. The set

$$\Psi_s^{iso} = \Phi_3^{iso} \cup \bigcup_{j=3}^{2+s} \Psi_j^{iso} \tag{4.15}$$

is a finite dimensional approximation of $\Psi^{iso}$.

## 5. Quantitative properties

In this section, we present the condition numbers of the stiffness matrices for the following elliptic problem:

$$-\epsilon \Delta u + a u = f \ \text{ on } \Omega, \quad u = 0 \text{ on } \partial\Omega, \tag{5.1}$$

where $\Delta$ is the Laplace operator, $\epsilon$ and $a$ are positive constants. The variational formulation for an anisotropic wavelet basis is

$$\mathbf{A}^{ani}\mathbf{u}^{ani} = \mathbf{f}^{ani}, \tag{5.2}$$

where

$$\mathbf{A}^{ani} := \epsilon \left\langle \nabla \Psi^{ani}, \nabla \Psi^{ani} \right\rangle + a \left\langle \Psi^{ani}, \Psi^{ani} \right\rangle, \tag{5.3}$$
$$u := \left(\mathbf{u}^{ani}\right)^T \Psi^{ani}, \quad \mathbf{f}^{ani} = \left\langle f, \Psi^{ani} \right\rangle.$$

An advantage of discretization of elliptic equation (5.1) using a wavelet basis is that the system (5.2) can be simply precondtioned by a diagonal preconditioner.[10] Let

$\mathbf{D}$ be a matrix of diagonal elements of the matrix $\mathbf{A}$, i.e. $\mathbf{D}_{\lambda,\mu} = \mathbf{A}_{\lambda,\mu}\delta_{\lambda,\mu}$, where $\delta_{\lambda,\mu}$ denotes Kronecker delta. Setting

$$\tilde{\mathbf{A}}^{ani} := \left(\mathbf{D}^{ani}\right)^{-1/2} \mathbf{A}^{ani} \left(\mathbf{D}^{ani}\right)^{-1/2}, \qquad (5.4)$$
$$\tilde{\mathbf{u}}^{ani} := \left(\mathbf{D}^{ani}\right)^{1/2} \mathbf{u}^{ani}, \quad \tilde{\mathbf{f}}^{ani} := \left(\mathbf{D}^{ani}\right)^{-1/2} \mathbf{f}^{ani}$$

we obtain the preconditioned system $\tilde{\mathbf{A}}^{ani}\tilde{\mathbf{u}}^{ani} = \tilde{\mathbf{f}}^{ani}$. It is known[10] that

$$\operatorname{cond} \tilde{\mathbf{A}}^{ani} \leq C < \infty. \qquad (5.5)$$

Let

$$\mathbf{A}_s^{ani} = \epsilon \left\langle \nabla\Psi_s^{ani}, \nabla\Psi_s^{ani} \right\rangle + a \left\langle \Psi_s^{ani}, \Psi_s^{ani} \right\rangle, \qquad (5.6)$$
$$u_s^{ani} = \left(\mathbf{u}_s^{ani}\right)^T \Psi_s^{ani}, \quad \mathbf{f}_s^{ani} = \left\langle f, \Psi_s^{ani} \right\rangle.$$

and let $\mathbf{D}_s^{ani}$ be a matrix of diagonal elements of the matrix $\mathbf{A}_s^{ani}$, i.e. $\left(\mathbf{D}_s^{ani}\right)_{\lambda,\mu} = \left(\mathbf{A}_s^{ani}\right)_{\lambda,\mu} \delta_{\lambda,\mu}$. We set

$$\tilde{\mathbf{A}}_s^{ani} := \left(\mathbf{D}_s^{ani}\right)^{-1/2} \mathbf{A}_s^{ani} \left(\mathbf{D}_s^{ani}\right)^{-1/2}, \qquad (5.7)$$
$$\tilde{\mathbf{u}}_s^{ani} := \left(\mathbf{D}_s^{ani}\right)^{1/2} \mathbf{u}_s^{ani}, \quad \tilde{\mathbf{f}}_s^{ani} := \left(\mathbf{D}_s^{ani}\right)^{-1/2} \mathbf{f}_s^{ani}$$

and obtain preconditioned finite-dimensional system

$$\tilde{\mathbf{A}}_s^{ani}\tilde{\mathbf{u}}_s^{ani} = \tilde{\mathbf{f}}_s^{ani}. \qquad (5.8)$$

Since $\tilde{\mathbf{A}}_s^{ani}$ is a part of the matrix $\mathbf{A}^{ani}$ that is symmetric and positive definite, we have also

$$\operatorname{cond} \tilde{\mathbf{A}}_s^{ani} \leq C. \qquad (5.9)$$

The preconditioned system for an isotropic wavelet basis

$$\tilde{\mathbf{A}}_s^{iso}\tilde{\mathbf{u}}_s^{iso} = \tilde{\mathbf{f}}_s^{iso}. \qquad (5.10)$$

is derived in a similar way. The stiffness matrix $\tilde{\mathbf{A}}_s^{iso}$ also satisfies

$$\operatorname{cond} \tilde{\mathbf{A}}_s^{iso} \leq C. \qquad (5.11)$$

The eigenvalues and condition numbers of the stiffness matrices for one-dimensional problem are shown in Table 1. We denote the stiffness matrix for the bases $\Psi_s$ and $\Psi_s^{ort}$ preconditioned as in (5.7) by $\tilde{\mathbf{A}}_s$ and $\tilde{\mathbf{A}}_s^{ort}$, respectively. The consequence of Remark 1.1 is that the condition number with respect to the $H^1$-seminorm of the multiscale wavelet basis $\Psi_s$ normalized with respect to the $H^1$-seminorm is equal to the square root of the condition number of the stiffness matrix $\tilde{\mathbf{A}}_s$. The eigenvalues and condition numbers of the stiffness matrices for two-dimensional and three-dimensional problems are shown in Table 2 and Table 3. Table 1, Table 2 and Table 3 correspond to the choice of parameters $\epsilon = 1$ and $a = 0$, i.e. for the Poisson equation.

In Table 4 and Table 5 a dependence of the condition number on the parameter $\epsilon$ is shown. It is computed for the two-dimensional problem and $a = 1$. It can be

Table 1. The maximal eigenvalues, the minimal eigenvalues and the condition numbers numbers of the matrices $\tilde{\mathbf{A}}_s^{ort}$ and $\tilde{\mathbf{A}}_s$ of the size $N \times N$ corresponding to the one-dimensional problem.

| $s$ | $N$ | $\lambda_{max}^{ort}$ | $\lambda_{min}^{ort}$ | $\mathrm{cond}\tilde{\mathbf{A}}_s^{ort}$ | $\lambda_{max}$ | $\lambda_{min}$ | $\mathrm{cond}\tilde{\mathbf{A}}_s$ |
|---|---|---|---|---|---|---|---|
| 1 | 17 | 4.03 | 5.56 | 2.89 | 19.83 | 7.74 | 2.89 |
| 2 | 33 | 5.55 | 5.60 | 2.78 | 21.35 | 7.79 | 2.78 |
| 3 | 65 | 6.85 | 5.61 | 2.78 | 22.13 | 7.81 | 2.78 |
| 4 | 129 | 8.00 | 5.62 | 2.78 | 22.66 | 7.81 | 2.78 |
| 5 | 257 | 9.01 | 5.62 | 2.78 | 23.03 | 7.82 | 2.78 |
| 6 | 513 | 9.91 | 5.62 | 2.78 | 23.30 | 7.82 | 2.78 |
| 7 | 1 025 | 10.68 | 5.62 | 2.78 | 23.49 | 7.82 | 2.78 |
| 8 | 2 049 | 11.35 | 5.62 | 2.78 | 23.63 | 7.82 | 2.78 |

Table 2. The maximal eigenvalues, the minimal eigenvalues and the condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{ani}$ and $\tilde{\mathbf{A}}_s^{iso}$ of the size $N \times N$ corresponding to the two-dimensional problem.

| $s$ | $N$ | $\lambda_{max}^{ani}$ | $\lambda_{min}^{ani}$ | $\mathrm{cond}\tilde{\mathbf{A}}_s^{ani}$ | $\lambda_{max}^{iso}$ | $\lambda_{min}^{iso}$ | $\mathrm{cond}\tilde{\mathbf{A}}_s^{iso}$ |
|---|---|---|---|---|---|---|---|
| 1 | 289 | 2.46 | 0.15 | 16.2 | 3.21 | 0.06 | 51.6 |
| 2 | 1 089 | 2.67 | 0.14 | 19.2 | 3.27 | 0.06 | 58.4 |
| 3 | 4 225 | 2.80 | 0.12 | 23.8 | 3.29 | 0.06 | 58.8 |
| 4 | 16 641 | 2.88 | 0.10 | 29.6 | 3.31 | 0.06 | 59.0 |
| 5 | 66 049 | 2.92 | 0.08 | 35.4 | 3.31 | 0.06 | 59.2 |
| 6 | 263 169 | 2.94 | 0.07 | 41.1 | 3.32 | 0.06 | 59.2 |
| 7 | 1 058 841 | 2.95 | 0.06 | 46.3 | 3.32 | 0.06 | 59.3 |
| 8 | 4 231 249 | 2.96 | 0.06 | 50.9 | 3.32 | 0.06 | 59.3 |

Table 3. The maximal eigenvalues, the minimal eigenvalues and the condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{ani}$ and $\tilde{\mathbf{A}}_s^{iso}$ of the size $N \times N$ corresponding to the three-dimensional problem.

| $s$ | $N$ | $\lambda_{max}^{ani}$ | $\lambda_{min}^{ani}$ | $\mathrm{cond}\tilde{\mathbf{A}}_s^{ani}$ | $\lambda_{max}^{iso}$ | $\lambda_{min}^{iso}$ | $\mathrm{cond}\tilde{\mathbf{A}}_s^{iso}$ |
|---|---|---|---|---|---|---|---|
| 1 | 4 913 | 3.94 | 0.07 | 58.2 | 6.34 | 0.01 | 829.3 |
| 2 | 35 937 | 4.47 | 0.05 | 88.0 | 6.47 | 0.01 | 871.4 |
| 3 | 274 625 | 4.77 | 0.04 | 125.4 | 6.52 | 0.01 | 879.5 |
| 4 | 2 146 689 | 5.01 | 0.03 | 181.2 | 6.56 | 0.01 | 883.0 |
| 5 | 16 974 593 | 5.12 | 0.02 | 250.7 | 6.56 | 0.01 | 885.0 |

seen that if $\epsilon$ increases the condition number become close to the condtion number of the stiffness matrix for the Poisson problem and if $\epsilon$ decreases than the condition number become close to the condition number of Grammian matrix with respect to the $L^2$-inner product, i.e. the case $\epsilon = 0$ and $a = 1$. The condition numbers are even significantly lower than condition numbers for one-dimensional problem and periodized biorthogonal wavelets, see Tables in Ref. 22.

Table 4. Condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{iso}$ of the size $N \times N$ for various values of $\epsilon$ corresponding to the two-dimensional problem .

| $s$ | $N$ | $\epsilon = 10^3$ | $\epsilon = 1$ | $\epsilon = 10^{-3}$ | $\epsilon = 10^{-9}$ | $\epsilon = 0$ |
|---|---|---|---|---|---|---|
| 1 | 289 | 51.6 | 51.6 | 145.3 | 393.1 | 393.1 |
| 2 | 1 089 | 58.4 | 58.4 | 146.7 | 447.8 | 447.8 |
| 3 | 4 225 | 58.8 | 58.8 | 146.8 | 471.3 | 471.4 |
| 4 | 16 641 | 59.0 | 59.0 | 146.8 | 484.0 | 484.0 |
| 5 | 66 049 | 59.2 | 59.2 | 146.8 | 491.1 | 491.1 |
| 6 | 263 169 | 59.2 | 59.2 | 146.8 | 494.8 | 494.9 |
| 7 | 1 058 841 | 59.3 | 59.3 | 146.8 | 496.8 | 496.9 |
| 8 | 4 231 249 | 59.3 | 59.3 | 146.8 | 497.8 | 497.9 |

Table 5. Condition numbers of the stiffness matrices $\tilde{\mathbf{A}}_s^{ani}$ of the size $N \times N$ for various values of $\epsilon$ corresponding to the two-dimensional problem .

| $s$ | $N$ | $\epsilon = 10^3$ | $\epsilon = 1$ | $\epsilon = 10^{-3}$ | $\epsilon = 10^{-9}$ | $\epsilon = 0$ |
|---|---|---|---|---|---|---|
| 1 | 289 | 16.2 | 16.2 | 15.1 | 16.2 | 16.2 |
| 2 | 1 089 | 19.2 | 19.2 | 19.0 | 30.8 | 30.8 |
| 3 | 4 225 | 23.8 | 23.8 | 23.5 | 46.9 | 46.9 |
| 4 | 16 641 | 29.6 | 29.6 | 29.4 | 63.9 | 63.9 |
| 5 | 66 049 | 35.6 | 35.5 | 35.4 | 81.2 | 81.3 |
| 6 | 263 169 | 41.3 | 41.1 | 41.1 | 98.0 | 98.1 |
| 7 | 1 058 841 | 46.4 | 46.3 | 46.3 | 113.6 | 113.9 |
| 8 | 4 231 249 | 51.0 | 51.0 | 51.0 | 127.2 | 128.9 |

## 6. Numerical example

The constructed wavelet basis can be used for solving various types of problems. Let us mention for example solving partial differential and integral equations by adaptive wavelet method.[6,7] In this section we use constructed wavelet basis in wavelet-Galerkin method. We consider the problem (5.1) with $\Omega = (0,1)^2$, $\epsilon = 1$ and $a = 0$. The right-hand side $f$ is such that the solution $u$ is given by:

$$u\left(x,y\right) = v\left(x\right)v\left(y\right), \quad v\left(x\right) = x\left(1 - e^{5x-5}\right). \tag{6.1}$$

We discretize the equation using Galerkin method and the isotropic wavelet basis constructed in this paper and we obtain discrete problem $\tilde{\mathbf{A}}_s^{iso}\tilde{\mathbf{u}}_s = \tilde{\mathbf{f}}_s^{iso}$. We solve it by conjugate gradient method using a simple multilevel approach:

1. Compute $\tilde{\mathbf{A}}_s^{iso}$ and $\tilde{\mathbf{f}}_s^{iso}$, choose $\mathbf{v}_0$ of the length $9^2$.
2. For $j = 0, \ldots, s$ find the solution $\tilde{\mathbf{u}}_j$ of the system $\tilde{\mathbf{A}}_j^{iso}\tilde{\mathbf{u}}_j = \tilde{\mathbf{f}}_j^{iso}$ by conjugate

gradient method with initial vector $\mathbf{v}_j$ defined for $j \geq 1$ by

$$(\mathbf{v}_j) = \begin{cases} \tilde{\mathbf{u}}_{j-1}, & i = 1, \ldots, k_j, \\ 0, & i = k_j, \ldots, k_{j+1}, \end{cases} \tag{6.2}$$

where $k_j = \left(2^{j+2} + 1\right)^2$.

A criterion $\|\mathbf{r}_j\| < \epsilon_j$, where $\mathbf{r}_j := \tilde{\mathbf{A}}_j^{iso} \tilde{\mathbf{u}}_j - \tilde{\mathbf{f}}_j^{iso}$, is used for terminating iterations of conjugate gradient (CG) method at level $j$. It is possible to choose smaller $\epsilon_j$ on coarser levels,[13] but in our case we choose $\epsilon_j$ constant for all levels, because other choices of $\epsilon_j$ did not lead to significantly smaller number of iterations in our experiments. The method for anisotropic wavelet basis is similar.

We denote the number of iterations on the level $j$ as $M_j$. It is known[22] that one CG iteration requires $\mathcal{O}(N)$ floating-point operations, where $N \times N$ is the size of the matrix. Therefore the number of operations needed to compute one CG iteration on the level $j$ requires about one quarter of operations needed to compute one CG iteration on the level $j+1$, we compute the total number of iterations by

$$M = \sum_{j=0}^{s} \frac{M_j}{4^{s-j}}. \tag{6.3}$$

The results are listed in Table 6 and Table 7. The residuum is denoted $\mathbf{r}_s$, $u$ is the exact solution of the given problem and $u_s$ is an approximate solution obtained by multilevel Galerkin method with $s$ levels of wavelets. It can be seen that the number of conjugate gradient iterations is quite small and that

$$\frac{\|u_s - u\|_\infty}{\|u_{s+1} - u\|_\infty} \approx \frac{\|u_s - u\|}{\|u_{s+1} - u\|} \approx \frac{1}{16}, \tag{6.4}$$

i.e. that order of convergence is 4. It confirms the theory.

Table 6. Number of iterations and error estimates for multilevel conjugate gradient method for isotropic wavelet basis.

| $s$ | $N$ | $M$ | $\|\mathbf{r}_s\|$ | $\|u_s - u\|_\infty$ | $\|u_s - u\|$ |
|---|---|---|---|---|---|
| 1 | 289 | 17.00 | 1.00e-6 | 1.02e-5 | 2.95e-6 |
| 2 | 1 089 | 17.06 | 1.51e-7 | 6.95e-7 | 2.49e-7 |
| 3 | 4 225 | 16.75 | 1.29e-8 | 4.83e-8 | 1.61e-8 |
| 4 | 16 641 | 15.31 | 1.78e-9 | 2.87e-9 | 9.92e-10 |
| 5 | 66 049 | 14.48 | 1.59e-10 | 1.79e-10 | 6.18e-11 |
| 6 | 263 169 | 12.77 | 3.21e-11 | 1.12e-11 | 3.77e-12 |
| 7 | 1 058 841 | 12.16 | 3.11e-12 | 1.38e-12 | 6.45e-13 |

Table 7. Number of iterations and error estimates for multilevel conjugate gradient method for anisotropic wavelet basis.

| $s$ | $N$ | $M$ | $\|\mathbf{r}_s\|$ | $\|u_s - u\|_\infty$ | $\|u_s - u\|$ |
|---|---|---|---|---|---|
| 1 | 289 | 9.25 | 8.15e-6 | 1.03e-5 | 2.97e-6 |
| 2 | 1 089 | 11.13 | 1.16e-6 | 7.10e-7 | 2.49e-7 |
| 3 | 4 225 | 11.42 | 1.33e-7 | 4.91e-8 | 1.62e-8 |
| 4 | 16 641 | 12.05 | 1.32e-8 | 2.90e-9 | 9.93e-10 |
| 5 | 66 049 | 12.14 | 1.31e-9 | 1.76e-10 | 6.20e-11 |
| 6 | 263 169 | 11.95 | 1.32e-10 | 1.14e-11 | 3.78e-12 |
| 7 | 1 058 841 | 11.98 | 1.46e-11 | 1.24e-12 | 6.01e-13 |

## Acknowledgments

The authors have been supported by the project "Modern numerical methods" No. 5844/16 financed by Technical University in Liberec. The authors would like to thank T. Šimková for her help with numerical experiments.

## References

1. D. Černá and V. Finěk, Construction of optimally conditioned cubic spline wavelets on the interval, *Adv. Comput. Math.* **34** (2011) 219–252.
2. D. Černá and V. Finěk, Cubic spline wavelets with complementary boundary conditions, *Appl. Math. Comput.* **219** (2012) 1853–1865.
3. D. Černá and V. Finěk, Quadratic spline wavelets with short support for fourth-order problems, to appear in *Result. Math.*,**66** (2014) 525–540.
4. D. Černá and V. Finěk, Cubic spline wavelets with short support for fourth-order problems, *Appl. Math. Comput.* **243** (2014) 44–56.
5. C.K. Chui and E. Quak, Wavelets on a bounded interval, in *Numerical Methods of Approximation Theory*, eds. D. Braess, D. and L.L. Schumaker), (Birkhäuser Verlag, 1992), 53–75.
6. A. Cohen, W. Dahmen and R. DeVore, Adaptive wavelet schemes for elliptic operator equations - convergence rates, *Math. Comput.* **70** (2001) 27–75.
7. A. Cohen, W. Dahmen and R. DeVore, Adaptive wavelet methods II - beyond the elliptic case, *Found. Math.* **2** (2002) 203–245.
8. W. Dahmen, Stability of multiscale transformations, *J. Fourier Anal. Appl.* **4** (1996) 341–362.
9. W. Dahmen, B. Han, R.Q. Jia, and A. Kunoth, Biorthogonal multiwavelets on the interval: cubic Hermite splines, *Constr. Approx.* **16** (2000) 221–259.
10. W. Dahmen and A. Kunoth, Multilevel preconditioning, *Numer. Math.* **63** (1992) 315–344.
11. W. Dahmen, A. Kunoth, and K. Urban, Biorthogonal spline wavelets on the interval - stability and moment conditions, *Appl. Comp. Harm. Anal.* **6** (1999) 132–196.
12. T.J. Dijkema and R. Stevenson, A sparse Laplacian in tensor product wavelet coordinates, *Numer. Math.* **115** (2010) 433–449.
13. R.Q. Jia and S.T. Liu, Wavelet bases of Hermite cubic splines on the interval, *Adv. Comput. Math* **25** (2006) 23–39.
14. R.Q. Jia, Spline wavelets on the interval with homogeneous boundary conditions, *Adv.*

*Comput. Math* **30** (2009) 177–200.
15. R.Q. Jia and W. Zhao, Riesz bases of wavelets and applications to numerical solutions of elliptic equations, *Math. Comput.* **80** (2011) 1525–1556.
16. R. Pabel, Wavelet methods for PDE constrained elliptic control problems with Dirichlet boundary control, Thesis, Universität Bonn (2005).
17. M. Primbs, New stable biorthogonal spline-wavelets on the interval, *Result. Math.* **57** (2010) 121–162.
18. S.M. Quraishi and K. Sandeep, New bi-cubic spline wavelet based thin plate finite elements, Int. J. Multiscale Comput. Eng., DOI: 10.1615/IntJMultCompEng.2014002616
19. S.M. Quraishi and K. Sandeep, New cubic Hermite finite element wavelets with improved properties, Int. J. Wavelets Multiresolut. Inf. Process.
20. A. Schneider, Biorthogonal cubic Hermite spline multiwavelets on the interval with complementary boundary conditions, *Result. Math.* **53** (2009) 407–416.
21. A. Tavakoli and F. Zarmehi, Construction of the matched multiple knot B-spline wavelets on a bounded interval, *Int. J. Computer Math.*, DOI: 10.1080/00207160.2014.960403
22. K. Urban, *Wavelet methods for elliptic partial differential equations*, Oxford University Press, Oxford, 2009.

**Results in Mathematics manuscript No.**
(will be inserted by the editor)

# On a sparse representation of a $n$-dimensional Laplacian in wavelet coordinates

**Dana Černá · Václav Finěk**

Received: date / Accepted: date

**Abstract** Important parts of adaptive wavelet methods are well conditioned wavelet stiffness matrices and an efficient approximate multiplication of quasi sparse stiffness matrices with vectors in wavelet coordinates. Therefore it is useful to develop a well conditioned wavelet basis with respect to which both the mass and stiffness matrices are sparse in the sense that the number of nonzero elements in any column is bounded by a constant. Consequently, the stiffness matrix corresponding to the $n$-dimensional Laplacian in tensor product wavelet basis is also sparse. Then, a matrix-vector multiplication can be performed exactly with linear complexity. In this paper, we construct a wavelet basis based on Hermite cubic splines with respect to which both the mass matrix and the stiffness matrix corresponding to one dimensional Poisson equation are sparse. Moreover, a proposed basis is very well conditioned on low decomposition levels. Small condition numbers for low decomposition levels and a sparse structure of stiffness matrices are kept for any well conditioned second order partial differential equations with constant coefficients and moreover they are independent of the space dimension.

**Keywords** Wavelet · Riesz bases · Cubic Hermite spline · Homogeneous Dirichlet boundary conditions · Condition numbers · Sparse representations

**Mathematics Subject Classification (2000)** 15A12 · 41A15 · 65F50 · 65N12 · 65T60

The authors have been supported by the SGS project "Numerical Methods II".

D. Černá
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec, Studentská 2, 461 17 Liberec, Czech Republic

V. Finěk
Department of Mathematics and Didactics of Mathematics, Technical University of Liberec, Studentská 2, 461 17 Liberec, Czech Republic
E-mail: Vaclav.Finek@tul.cz

## 1 Introduction

A general concept for solving of operator equations by means of wavelets was proposed by A. Cohen, W. Dahmen and R. DeVore in [8,9]. It consists of the following steps: transformation of the variational formulation into the well conditioned infinite-dimensional problem in the space $l^2$, finding of the convergent iteration process for the $l^2-$ problem and finally a derivation of its computable version. The aim is to find an approximation of the unknown solution $u$ which should correspond to the best $N$-term approximation, and the associated computational work should be proportional to the number of unknowns. Essential components to achieve this goal are well conditioned wavelet stiffness matrices and an efficient approximate multiplication of quasi-sparse wavelet stiffness matrices with vectors.

In [8], authors exploited an off-diagonal decay of entries of the wavelet stiffness matrices and designed a numerical routine **APPLY** which approximates the exact matrix-vector product with the desired tolerance $\epsilon$ and that has linear computational complexity, up to sorting operations. The idea of **APPLY** is following: To truncate $\mathbf{A}$ in scale by zeroing $a_{i,j}$ whenever $\delta(i,j) > k$ ($\delta$ represents the level difference of two functions in the wavelet expansion) and denote resulting matrix by $\mathbf{A_k}$. At the same time to sort vector entries $\mathbf{v}$ with respect to the size of their absolute values. One obtains $\mathbf{v_k}$ by retaining $2^k$ biggest coefficients in absolute values of $\mathbf{v}$ and setting all other equal to zero. The maximum value of $k$ should be determined to reach a desired accuracy of approximation. Then one computes an approximation of $\mathbf{Av}$ by

$$\mathbf{w} := \mathbf{A_k v_0} + \mathbf{A_{k-1}}(\mathbf{v_1} - \mathbf{v_0}) + \ldots + \mathbf{A_0}(\mathbf{v_k} - \mathbf{v_{k-1}}) \qquad (1)$$

with the aim to balance both accuracy and computational complexity at the same time. In [16], binning and approximate sorting strategy was used to eliminate these sorting costs and then an asymptotically optimal algorithm was obtained. The idea is following: Reorder the elements of $\mathbf{v}$ into the sets $V_0, \ldots, V_q$, where $v_\lambda \in V_i$ if and only if

$$2^{-i-1} \|\mathbf{v}\|_{l^2} < v_\lambda < 2^{-i} \|\mathbf{v}\|_{l^2}, \quad 0 \le i < q.$$

Eventual remaining elements are put into the set $V_q$. And subsequently to generate vectors $\mathbf{v_k}$ by successively extracting $2^k$ elements from $\bigcup_i V_i$, starting from $V_0$ and when it is empty continuing with $V_1$ and so forth. Finally the scheme (1) is applied. Further improvements of this scheme were proposed in [4,12]. Although the **APPLY** routine has optimal computational complexity, its application is relatively time consuming and moreover it is not easy to implement it efficiently.

It is well known, that condition numbers of stiffness matrices in wavelet coordinates depend on Riesz constants of a wavelet basis. Before we explain it in more detail, we start with a definition of a wavelet basis. We consider here families $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} \subset L_2(0,1)$ of functions (wavelets) where $\mathcal{J}$ is an infinite index set and $\mathcal{J} = \mathcal{J}_\Phi \cup \mathcal{J}_\Psi$, where $\mathcal{J}_\Phi$ is a finite set representing scaling functions living on the coarsest scale. Any index $\lambda \in \mathcal{J}$ is of the form $\lambda = (j,k)$, where $|\lambda| = j$ denotes a scale and $k$ denotes spatial location. The above notation enables us to write wavelet expansions as

$$\mathbf{d}^T \Psi := \sum_{\lambda \in \mathcal{J}} d_\lambda \psi_\lambda.$$

At last, for $s \geq 0$ the space $H^s$ will denote a closed subspace of the Sobolev space $H^s(0,1)$, defined e.g. by imposing homogeneous boundary conditions at one or both endpoints, and for $s < 0$ the space $H^s$ will denote the dual space $H^s := (H^{-s})'$. $\|.\|_{H^s}$ will denote the corresponding norm. Further $l_2(\mathcal{J})$ will denote the space consisting of the power summable sequences and $\|.\|_{l_2(\mathcal{J})}$ will denote the corresponding norm.

A family $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\} \subset L_2(0,1)$ is called a *wavelet basis* of $H^s$ for some $\gamma, \widetilde{\gamma} > 0$ and $s \in (-\widetilde{\gamma}, \gamma)$, if

– $\Psi$ normalized in $H^s$ is a Riesz basis of $H^s$, that means $\Psi$ forms a basis of $H^s$ and there exist constants $c_s, C_s > 0$ such that for all $\mathbf{b} = \{b_\lambda\}_{\lambda \in \mathcal{J}} \in l_2(\mathcal{J})$ holds
$$c_s \|\mathbf{b}\|_{l_2(\mathcal{J})} \leq \left\| \sum_{\lambda \in \mathcal{J}} \frac{b_\lambda \psi_\lambda}{\|\psi_\lambda\|_{H^s}} \right\|_{H^s} \leq C_s \|\mathbf{b}\|_{l_2(\mathcal{J})}, \qquad (2)$$
where $\sup c_s$, $\inf C_s$ are called Riesz bounds and $\operatorname{cond}(\Psi) := \dfrac{\inf C_s}{\sup c_s}$ is called the condition number of $\Psi$.
– Functions are local in the sense that $\operatorname{diam}(\operatorname{supp} \psi_\lambda) \leq C 2^{-|\lambda|}$ for all $\lambda \in \mathcal{J}$, where $C$ is a constant independent of $\lambda$.
– Functions $\psi_\lambda$, $\lambda \in \mathcal{J}_\Psi$, have cancellation properties of order $m$, i.e.
$$\left| \int_0^1 v(x)\, \psi_\lambda(x)\, dx \right| \leq 2^{-m|\lambda|} |v|_{H^m(0,1)}, \qquad \forall v \in H^m(0,1).$$
It means that integration against wavelets eliminates smooth parts of functions and it is equivalent with vanishing wavelet moments of order $m$.

We consider here the following Dirichlet problem

$$u - \sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = f \quad \text{in} \quad \Omega = (0,1)^d \qquad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega \qquad (3)$$

for given $f \in H^{-1}(\Omega)$. A Riesz wavelet basis for $H_0^1(\Omega)$ can be constructed by a tensor product of univariate Riesz wavelet bases. Indeed, let $\Psi = \{\psi_\lambda, \lambda \in \mathcal{J}\}$ be after appropriate normalization a Riesz wavelet basis for spaces $L_2(0,1)$ and $H_0^1(0,1)$ then

$$\boldsymbol{\Psi} = \left\{ \boldsymbol{\psi}_{\boldsymbol{\lambda}} := \frac{\otimes_{j=1}^d \psi_{\lambda_m}}{\left\| \otimes_{j=1}^d \psi_{\lambda_m} \right\|_{H^1(\Omega)}}, \boldsymbol{\lambda} \in \mathcal{J}^d \right\} \qquad (4)$$

is a Riesz basis for $H_0^1(\Omega)$ (see [14]) with the Riesz constants (see [12])

$$\min(c_0,c_1)\, c_0^{d-1}\, \|\boldsymbol{b}\|^2_{l_2(\mathcal{J}^d)} \le \left\| \sum_{\boldsymbol{\lambda}\in\mathcal{J}^d} b_{\boldsymbol{\lambda}} \boldsymbol{\psi}_{\boldsymbol{\lambda}} \right\|^2_{H^1(\Omega)} \le \max(C_0,C_1)\, C_0^{d-1}\, \|\boldsymbol{b}\|^2_{l_2(\mathcal{J}^d)} \tag{5}$$

$\forall \boldsymbol{b} \in l_2\left(\mathcal{J}^d\right)$, where constants $c_0$, $C_0$, $c_1$, $C_1$ are Riesz constants with respect to spaces $L_2$ and $H_1$, respectively. Writing

$$u = \boldsymbol{u}^T \boldsymbol{\Psi} := \sum_{\boldsymbol{\lambda}\in\mathcal{J}^d} \boldsymbol{u}_{\boldsymbol{\lambda}} \boldsymbol{\psi}_{\boldsymbol{\lambda}} \qquad \text{and} \qquad \boldsymbol{f} = [f(\boldsymbol{\psi}_{\boldsymbol{\lambda}})]_{\boldsymbol{\lambda}\in\mathcal{J}^d},$$

then an equivalent formulation of (3) is

$$\boldsymbol{A}\boldsymbol{u} = \boldsymbol{f} \tag{6}$$

with

$$\boldsymbol{A} = \boldsymbol{D}^{-1}\left(\boldsymbol{M}\otimes\ldots\otimes\boldsymbol{M} + \boldsymbol{S}\otimes\ldots\otimes\boldsymbol{M} + \cdots + \boldsymbol{M}\otimes\ldots\otimes\boldsymbol{S}\right),$$

where $\boldsymbol{D} = \mathrm{diag}\,\left[\left\|\otimes_{j=1}^d \psi_{\lambda_m}\right\|_{H^1(\Omega)}\right]_{\boldsymbol{\lambda}\in\mathcal{J}^d}$, and

$$\mathbf{S} = \left[\int_0^1 \frac{\partial\psi_\lambda}{\partial x}\frac{\partial\psi_\mu}{\partial x}\, dx\right]_{\lambda,\mu\in\mathcal{J}} \qquad \text{and} \qquad \mathbf{M} = \left[\int_0^1 \psi_\lambda\, \psi_\mu\, dx\right]_{\lambda,\mu\in\mathcal{J}} \tag{7}$$

are the one-dimensional stiffness and the mass matrices, respectively. Then (5) implies

$$\mathrm{cond}\,(\boldsymbol{A}) \le \frac{\max(C_0,C_1)\, C_0^{d-1}}{\min(c_0,c_1)\, c_0^{d-1}}.$$

In general case, let us assume, that we have the following variational problem: for given $f\in\mathcal{H}'$ find $u\in\mathcal{H}$ such that

$$a(u,v) = f(v) \qquad \forall v\in\mathcal{H}, \tag{8}$$

where $\mathcal{H}$ is a Hilbert space and $a$ is a continuous bilinear form. Then, we define the operator $\mathcal{A}:\mathcal{H}\to\mathcal{H}'$ by

$$\mathcal{A}(u)(v) = a(u,v) \qquad \forall v\in\mathcal{H},$$

and then (8) is equivalent to

$$\mathcal{A}(u) = f. \tag{9}$$

If $a$ is $\mathcal{H}-$elliptic, then there exist positive constants $c_{\mathcal{A}}$, $C_{\mathcal{A}}$ such that

$$c_{\mathcal{A}}\, \|v\|_{\mathcal{H}} \le \|\mathcal{A}(v)\|_{\mathcal{H}'} \le C_{\mathcal{A}}\, \|v\|_{\mathcal{H}} \qquad \forall v\in\mathcal{H}. \tag{10}$$

Moreover, we will assume that we have a suitable wavelet basis $\mathbf{\Psi}$ of the space $\mathcal{H}$ normalized in $\mathcal{H}$ with Riesz constants $c$, $C$ and we define $\mathbf{A} = a(\mathbf{\Psi},\mathbf{\Psi})$ and $\mathbf{f} = f(\mathbf{\Psi})$, then

$$\mathcal{A}(u) = f \qquad \Longleftrightarrow \qquad \mathbf{A}\mathbf{u} = \mathbf{f},$$

where $u = \mathbf{u}^T \mathbf{\Psi}$, and

$$\operatorname{cond}(\mathbf{A}) \le \frac{C^2 C_{\mathcal{A}}}{c^2 c_{\mathcal{A}}}.$$

Proof can be found in [1].

Thus we can conclude that it is useful to develop well conditioned wavelet bases on the interval. Well conditioned wavelet basis for different types of wavelets and for different types of boundary conditions were already constructed in [2,3,5,6,15]. In this paper, we construct a wavelet basis based on Hermite cubic splines with respect to which both the mass matrix and the stiffness matrix corresponding to one dimensional Poisson equation are sparse. This means that the number of nonzero elements in any column is bounded independently of matrix size while stiffness matrices in wavelet coordinates are usually only quasi sparse. Then, matrix-vector multiplication can be performed exactly with linear complexity for any second order PDEs with constant coefficients. Moreover, the proposed basis is very well conditioned for low decomposition levels. Small condition numbers for low decomposition levels and a sparse structure of stiffness matrices are kept for any second order PDEs with constant coefficients, which are well conditioned in the sense of (10), and moreover they are independent of the space dimension. Wavelets with similar properties were already proposed in [13]. Our wavelets generate the same multiresolution spaces as wavelets from [13] but have improved condition numbers.

The paper is organized as follows: in the second section, we describe our construction, in the third section, we prove that the constructed basis is a Riesz basis and in the last section, we present condition numbers for model problems and compare them with condition numbers for a similar wavelet basis proposed in [13].

## 2 Cubic Hermite multiwavelets

We start with Hermite cubic splines as the primal scaling bases on interval. They are defined by

$$\phi_1(x) = \begin{cases} (x+1)^2(1-2x) & -1 \le x \le 0 \\ (1-x)^2(2x+1) & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}, \quad \phi_2(x) = \begin{cases} (x+1)^2 x & -1 \le x \le 0 \\ (1-x)^2 x & 0 \le x \le 1 \\ 0 & \text{otherwise} \end{cases}$$

or as a solutions of following scaling equations:

$$\phi_1(x) = \frac{1}{2}\phi_1(2x+1) + \phi_1(2x) + \frac{1}{2}\phi_1(2x-1) + \frac{3}{4}\phi_2(2x+1) - \frac{3}{4}\phi_2(2x-1),$$

$$\phi_2(x) = -\frac{1}{8}\phi_1(2x+1) + \frac{1}{8}\phi_1(2x-1) - \frac{1}{8}\phi_2(2x+1) + \frac{1}{2}\phi_2(2x) - \frac{1}{8}\phi_2(2x-1).$$

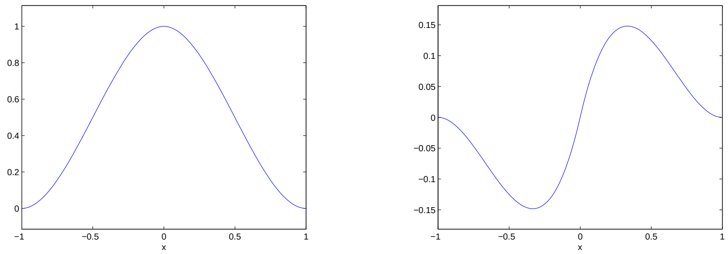



**Fig. 1** The Hermite cubic splines $\phi_1$ and $\phi_2$.

For $n \geq 1$, let $V_n$ be the space of piecewise cubic splines $v \in C^1(0,1) \cap C[0,1]$ for which $v(0) = v(1) = 0$. The dimension of $V_n$ is $2^{n+1}$ and the set

$$\Phi_n := \{\phi_1(2^n x - j) : j = 1, ..., 2^n - 1\} \cup \left\{\phi_2(2^n x - j)|_{[0,1]} : j = 0, ..., 2^n\right\}$$

is a basis for $V_n$. Let $W_n$ be the complement space of $V_n$ in $V_{n+1}$ then we have the following decomposition of space $H_0^1(0,1)$

$$H_0^1(0,1) = V_1 \oplus W_1 \oplus W_2 \oplus W_3 \oplus \ldots .$$

We construct four wavelets in such a way that wavelets from the space $W_{n+1}$ are orthogonal to the scaling functions from the space $V_n$ for $n \geq 1$. This property ensures that both the mass and stiffness matrices corresponding to the one-dimensional Laplacian have at most three wavelet blocks of nonzero elements in any column and then the number of nonzero elements in any column is bounded independently of matrix size. The first two wavelets have supports in $[-1, 1]$ and are uniquely determined by their orthogonality to cubic polynomials and by imposing that the first one is odd and the second one is even:

$$\psi_1(x) = \phi_1(2x+1) - \phi_1(2x-1) + \frac{39}{7}\phi_2(2x+1) + \frac{132}{7}\phi_2(2x) + \frac{39}{7}\phi_2(2x-1),$$

$$\psi_2(x) = -\frac{1}{2}\phi_1(2x+1) + \phi_1(2x) - \frac{1}{2}\phi_1(2x-1) - \frac{15}{4}\phi_2(2x+1) + \frac{15}{4}\phi_2(2x-1).$$

The second two wavelets have supports in $[-2, 2]$. And we impose on them again the above orthogonality condition which will be ensured by requiring that they are orthogonal to cubic polynomials on intervals $[-2, 0]$ and $[0, 2]$, respectively. Again one of them should be odd and the second one even. There remains several free parameters. To obtain a more sparse stiffness matrix and a better conditioned wavelet basis, we use these free parameters to prescribe the orthogonality of the first derivative of constructed wavelets to the first derivative of the first two wavelets. We obtain these two wavelets:

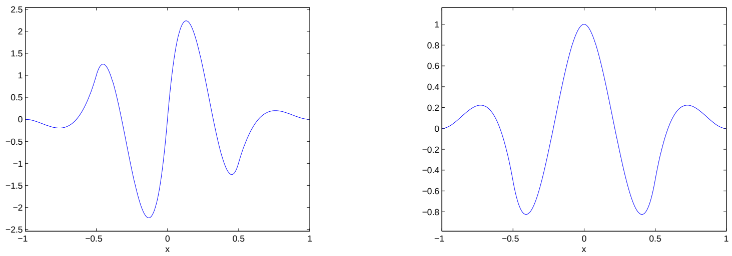



**Fig. 2** The first two wavelets $\psi_1$ and $\psi_2$.

$$\begin{aligned}\psi_3(x) = &-\frac{151}{480}\phi_1(2x+3)+\frac{2}{5}\phi_1(2x+2)-\frac{281}{480}\phi_1(2x+1)+\phi_1(2x)\\ &-\frac{281}{480}\phi_1(2x-1)+\frac{2}{5}\phi_1(2x-2)-\frac{151}{480}\phi_1(2x-3)\\ &-\frac{711}{224}\phi_2(2x+3)+\frac{79}{56}\phi_2(2x+2)-\frac{1551}{224}\phi_2(2x+1)\\ &+\frac{1551}{224}\phi_2(2x-1)-\frac{79}{56}\phi_2(2x-2)+\frac{711}{224}\phi_2(2x-3),\end{aligned}$$

$$\begin{aligned}\psi_4(x) = &\frac{7}{40}\phi_1(2x+3)-\frac{19}{90}\phi_1(2x+2)+\frac{163}{360}\phi_1(2x+1)\\ &-\frac{163}{360}\phi_1(2x-1)+\frac{19}{90}\phi_1(2x-2)-\frac{7}{40}\phi_1(2x-3)\\ &\frac{12}{7}\phi_2(2x+3)-\frac{25}{42}\phi_2(2x+2)+\frac{33}{7}\phi_2(2x+1)+5\phi_2(2x)\\ &+\frac{33}{7}\phi_2(2x-1)-\frac{25}{42}\phi_2(2x-2)+\frac{12}{7}\phi_2(2x-3),\end{aligned}$$

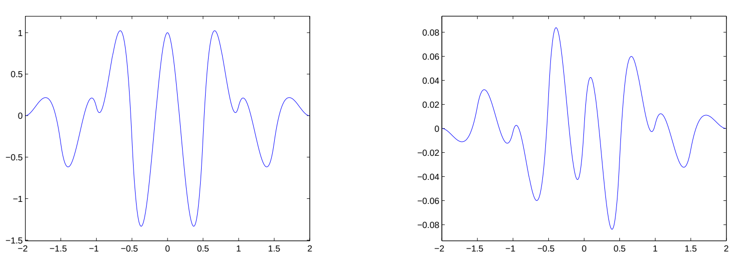



**Fig. 3** The second two wavelets $\psi_3$ and $\psi_4$.

Then a basis of the space $W_n$ is defined by

$$\begin{aligned}\Psi_n := &\left\{2^{\frac{n+1}{2}}\psi_1(2^n x-2j-1),\, 2^{\frac{n+1}{2}}\psi_2(2^n x-2j-1) : j=0,...,2^{n-1}-1\right\}\\ &\cup\left\{2^{\frac{n+1}{2}}\psi_3(2^n x-2j) : j=1,...,2^{n-1}-1\right\}\\ &\cup\left\{2^{\frac{n+1}{2}}\psi_4(2^n x-2j)|_{[0,1]} : j=0,...,2^{n-1}\right\}.\end{aligned}$$

Now, we would like to improve condition numbers of the constructed wavelet basis and to preserve or improve a sparse structure of the stiffness matrix corresponding to the one-dimensional Laplacian and a sparser structure of the mass matrix, respectively. We modify boundary scaling functions at the coarsest level and also wavelets at the coarsest level. A span of new functions will be the same as a span of original functions. First, we modify both boundary scaling functions $\phi_2(2x)|_{[0,1]}$ and $\phi_2(2x-2)|_{[0,1]}$ at the coarsest level in such a way that new boundary functions will be orthogonal to functions $\phi_1(2x-1)$ and $\phi_2(2x-1)$, and moreover they also will be also mutually orthogonal. And we obtain

$$\begin{aligned}\phi_3(x) &= \frac{4}{3}\phi_2(x+1)|_{[-1,1]} + \phi_2(x) + \frac{4}{3}\phi_2(x-1)|_{[-1,1]},\\ \phi_4(x) &= -12\phi_2(x+1)|_{[-1,1]} + \phi_1(x) + 12\phi_2(x-1)|_{[-1,1]}.\end{aligned}$$

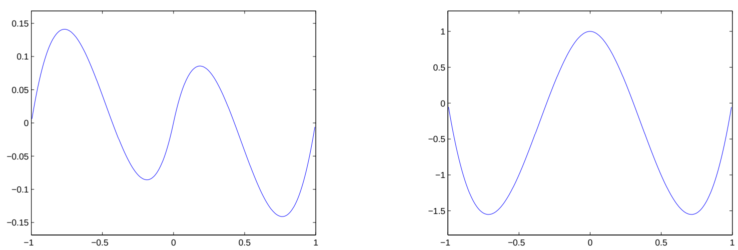


**Fig. 4** The modified boundary scaling functions $\phi_3$ and $\phi_4$.

Now a basis of the space $V_1$ is defined by

$$\widehat{\Phi}_1 := \{\phi_1(2x-1),\, \phi_2(2x-1),\, \phi_3(2x-1),\, \phi_4(2x-1)\}.$$

To further improve condition numbers of the constructed basis, we construct new basis functions for the space $W_1$. The first two wavelets will be orthogonal to scaling functions from the space $V_1$, will not depend on the boundary scaling functions from the space $V_2$ and one of them will be odd and the second one even. We obtain these two wavelets:

$$\psi_5(x) = \phi_1(2x+1) - \frac{794}{331}\phi_1(2x) + \phi_1(2x-1) + \frac{8793}{662}\phi_2(2x+1) - \frac{8793}{662}\phi_2(2x-1),$$

$$\psi_6(x) = -\phi_1(2x+1)+\phi_1(2x-1)-\frac{143}{15}\phi_2(2x+1)-\frac{52}{3}\phi_2(2x)-\frac{143}{15}\phi_2(2x-1).$$

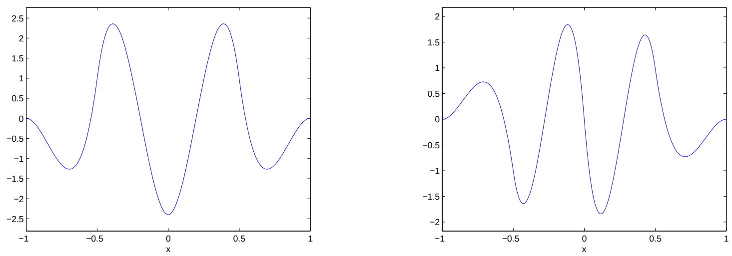


**Fig. 5** The first two modified wavelets $\psi_5$ and $\psi_6$.

The second two wavelets will be again orthogonal to scaling functions from the space $V_1$ and moreover will be orthogonal to the first two newly constructed wavelets. Again one of them will be odd and the second one even. Then we obtain:

$$\psi_7(x) = \phi_1(2x+1) - \phi_1(2x-1) - \frac{144}{7}\phi_2(2x+2)|_{[-1,1]} - \frac{275}{21}\phi_2(2x+1) \\ -\frac{68}{21}\phi_2(2x) - \frac{275}{21}\phi_2(2x-1) - \frac{144}{7}\phi_2(2x-2)|_{[-1,1]}.$$

$$\psi_8(x) = \frac{6947}{32022}\left(\phi_1(2x+1)+\phi_1(2x-1)\right) + \frac{\phi_1(2x)}{54} - \frac{2137}{593}\phi_2(2x+2)|_{[-1,1]} \\ -\frac{25327}{14232}\phi_2(2x+1) + \frac{25327}{14232}\phi_2(2x-1) + \frac{2137}{593}\phi_2(2x-2)|_{[-1,1]},$$

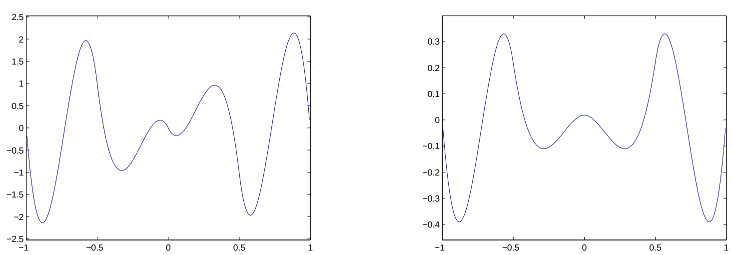


**Fig. 6** The second two modified wavelets $\psi_7$ and $\psi_8$.

Now a basis of the space $W_1$ is defined by

$$\widehat{\Psi}_1 := \{\psi_5(2x-1),\, \psi_6(2x-1),\, \psi_7(2x-1),\, \psi_8(2x-1)\}$$

and a basis of the space $H_0^1(0,1)$ is defined by

$$\Psi = \widehat{\Phi}_1 \cup \widehat{\Psi}_1 \bigcup_{j=2}^{\infty} \Psi_j. \tag{11}$$

A sparsity patterns for the mass matrix $\mathbf{M}$ and for the one dimensional stiffness matrix $\mathbf{S}$, respectively, defined in (7) can be seen in Figure 2. In the next section, we prove that it is a wavelet basis.

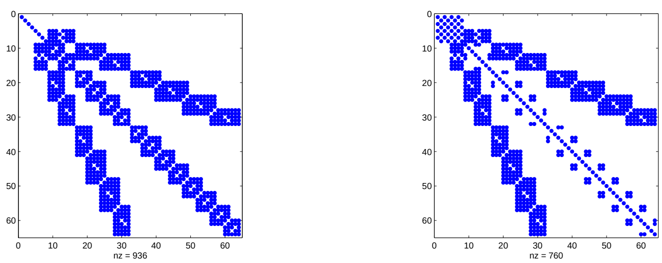



**Fig. 7** Nonzero elements of the mass matrix $\mathbf{M}$ and the stiffness matrix $\mathbf{S}$.

## 3 Properties of the constructed basis

To proof that the constructed basis forms a Riesz basis of the space $H_0^1(0,1)$, we use the following theorem from [13] which summarizes results from [10,11]:

**Theorem 1** *Let $j_0$ be the coarsest level and let*

$$V_{j_0} \subset V_{j_0+1} \subset \ldots \subset L_2(0,1), \qquad \widetilde{V}_{j_0} \subset \widetilde{V}_{j_0+1} \subset \ldots \subset L_2(0,1)$$

*be sequences of primal and dual spaces with*

$$\dim V_j = \dim \widetilde{V}_j$$

*such that for uniform $L_2(0,1)-$Riesz bases $\Phi_j$ and $\widetilde{\Phi}_j$ for $V_j$ and $\widetilde{V}_j$, respectively,*

$$\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)}^{-1}$$

*exists with a uniformly bounded spectral norm. In addition, for some $0 < \gamma < d$, let (Jackson or direct estimate)*

$$\inf_{v_j \in V_j} ||v - v_j||_{L_2(0,1)} \lesssim 2^{-jd} ||v||_{\mathcal{H}^d(0,1)} \qquad \forall v \in \mathcal{H}^d(0,1),$$

*and (Bernstein or inverse estimate)*

$$||v_j||_{\mathcal{H}^s(0,1)} \lesssim 2^{js} ||v_j||_{L_2(0,1)} \qquad \forall v_j \in V_j,\ s \in [0,\gamma),$$

*where, for* $s \in [0,d]$, $\mathcal{H}^s(0,1) = \left[L_2(0,1), H^d(0,1) \cap H_0^1(0,1)\right]_{s/d}$, *and let similar estimates be valid at the dual side with* $V_j, d, \gamma, \mathcal{H}^s(0,1)$ *reading as* $\widetilde{V}_j, \widetilde{d}, \widetilde{\gamma}, \widetilde{\mathcal{H}}^s(0,1)$. *And let* $\Psi_j$ *be uniform* $L_2(0,1)-$*Riesz bases for* $W_j := V_{j+1} \cap \widetilde{V}_j^{\perp L_2(0,1)}$, *then for* $s \in (-\widetilde{\gamma}, \gamma)$ *the collection*

$$\Phi_{j_0} \cup \bigcup_{j \in \mathbb{N}} 2^{-sj} \Psi_{j_0+j}$$

*is a Riesz basis for* $\mathcal{H}^s(0,1)$, *where* $\mathcal{H}^s(0,1) := (\mathcal{H}^{-s}(0,1))'$ *for* $s < 0$.

$C \lesssim D$ means that $C$ can be bounded by a multiple of $D$ independently of parameters on which they may depend. Let $\widetilde{V}_1 := V_1$ and let the basis for both spaces be

$$\widetilde{\Phi}_1 = \left\{ \sqrt{\frac{35}{13}} \phi_1(2x-1), \sqrt{105}\, \phi_2(2x-1), \sqrt{135}\, \phi_3(2x-1), \phi_4(2x-1) \right\}.$$

Then

$$\left\langle \widetilde{\Phi}_1, \widetilde{\Phi}_1 \right\rangle_{L_2(0,1)} = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}.$$

And for $j > 1$, let $\widetilde{V}_j$ be the space of piecewise cubic functions on intervals $\left[k2^{-j+1}, (k+1)2^{-j+1}\right]$ for $k = 0, \ldots, 2^{j-1} - 1$. Then dimension of $\widetilde{V}_j$ is apparently $2^{j+1}$ and from the construction immediately follows that $W_j = V_{j+1} \cap \widetilde{V}_j^{\perp L_2(0,1)}$. Now, we construct uniform $L_2(0,1)-$Riesz bases $\Phi_j$ and $\widetilde{\Phi}_j$ for spaces $V_j$, and $\widetilde{V}_j$, respectively, such that $\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)}^{-1}$ exists with a uniformly bounded spectral norm. It means that Riesz bounds are independent of $j$.

**Theorem 2** *There exists uniform* $L_2(0,1)-$*Riesz bases* $\Phi_j$ *and* $\widetilde{\Phi}_j$ *for* $V_j$ *and* $\widetilde{V}_j$, *respectively, such that* $\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)}^{-1}$ *exists with a uniformly bounded spectral norm.*

**Proof.** We start with functions $\phi_1(2x)$, $\phi_2(2x)$, $\phi_1(2x-1)$, and $\phi_2(2x-1)$ which span the space of $C^1(0,1)$ cubic splines on the interval $[0,1]$ and with functions $\widetilde{\phi}_i(\cdot) := (x - 1/2)^i|_{[0,1]}$ for $i = 0,1,2,3$ which span the space of piecewise cubic functions on $[0,1]$. Further we apply a number of transformation at the both bases to obtain a sparse and strictly diagonally dominant matrices $\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)}$. We keep the functions

$$\alpha_1(2x-1) := \phi_1(2x-1) \qquad \text{and} \qquad \alpha_2(2x-1) := 6\phi_2(2x-1)$$

which are supported in the interval $[0,1]$.

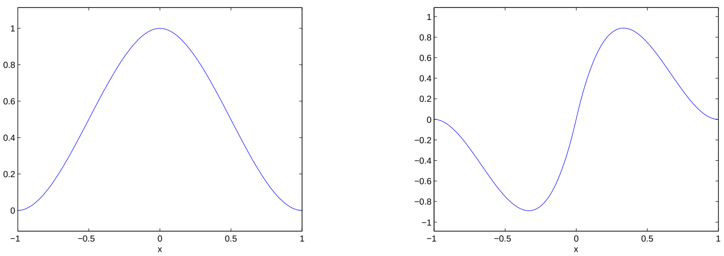



**Fig. 8** The first two primal basis functions $\alpha_1$ and $\alpha_2$.

In the second step, we construct the first two dual basis functions in a such a way that these new dual basis functions are orthogonal to the first two primal basis functions. Moreover the first new dual function is a linear combination of even polynomials while the second one is a linear combination of odd polynomials. After appropriate normalization, we obtain

$$\beta_1(x) := \left(3 - 30\left(x - \frac{1}{2}\right)^2\right)\Big|_{[0,1]}$$

and

$$\beta_2(x) := \left(\frac{75}{4}\left(x - \frac{1}{2}\right) - \frac{245}{2}\left(x - \frac{1}{2}\right)^3\right)\Big|_{[0,1]}.$$

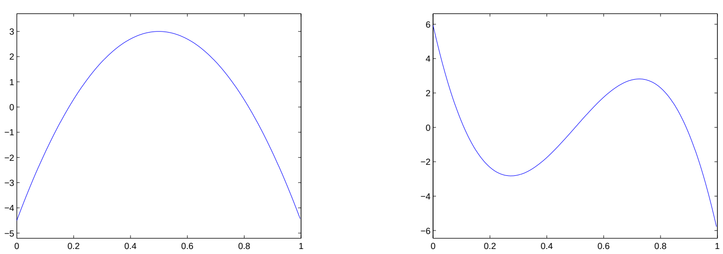



**Fig. 9** The first two dual basis functions $\beta_1$ and $\beta_2$.

In the third step, we construct the second two new primal basis functions as a linear combination of functions $\phi_1(2x)$, $\phi_2(2x)$, $\phi_1(2x-1)$, $\phi_2(2x-1)$, $\phi_1(2x-2)$, and $\phi_2(2x-2)$ in such a way that these new primal basis functions are orthogonal to dual functions $\beta_1(x)$, $\beta_2(x)$, $\beta_1(x-1)$, and $\beta_2(x-1)$. Moreover, we require that the first new primal basis function is even with respect to the point $x = 1$, and the second one is odd with respect to the point $x = 1$. We obtain

$$\alpha_3(2x) := \phi_1(2x) + 4\phi_1(2x-1) + \phi_1(2x-2)$$

and

$$\alpha_4(2x) := \phi_2(2x) + \frac{16}{5}\phi_2(2x-1) + \phi_2(2x-2).$$

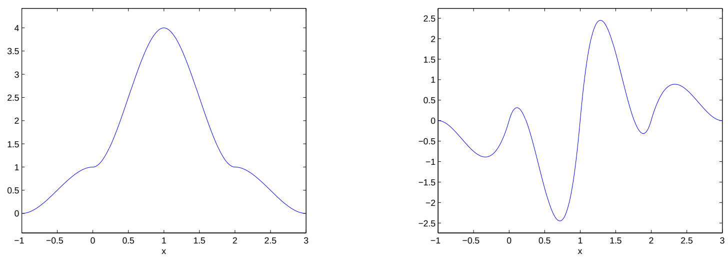



**Fig. 10** The second two primal basis functions $\alpha_3$ and $\alpha_4$.

In the fourth step, we construct the second two new dual basis functions as a linear combination of functions $\widetilde{\phi}_i(x)$ and $\widetilde{\phi}_i(x+1)$ for $i = 0, 1, 2, 3$ in such a way that these functions are orthogonal to functions $\alpha_1(2x-1)$, $\alpha_2(2x-1)$, $\alpha_1(2x+1)$, and $\alpha_2(2x+1)$. Moreover, we require that the first new dual basis function is orthogonal to functions $\alpha_4(2x-1)$, $\alpha_4(2x+1)$, and $\alpha_4(2x+3)$ and the second one is orthogonal to functions $\alpha_3(2x-1)$, $\alpha_3(2x+1)$, and $\alpha_3(2x+3)$. After appropriate normalization, we obtain

$$\begin{aligned}\beta_3(x) : = & \frac{-1}{14}\left(1 + 10\left(x+\frac{1}{2}\right) - 30\left(x+\frac{1}{2}\right)^2 - 140\left(x+\frac{1}{2}\right)^3\right)\Bigg|_{[-1,0]} \\ & - \frac{1}{14}\left(1 - 10\left(x-\frac{1}{2}\right) - 30\left(x-\frac{1}{2}\right)^2 + 140\left(x-\frac{1}{2}\right)^3\right)\Bigg|_{[0,1]}\end{aligned}$$

and

$$\begin{aligned}\beta_4(x) : = & \frac{5}{28}\left(1 + \frac{15}{2}\left(x+\frac{1}{2}\right) - 30\left(x+\frac{1}{2}\right)^2 - 105\left(x+\frac{1}{2}\right)^3\right)\Bigg|_{[-1,0]} \\ & + \frac{5}{28}\left(-1 + \frac{15}{2}\left(x-\frac{1}{2}\right) + 30\left(x-\frac{1}{2}\right)^2 - 105\left(x-\frac{1}{2}\right)^3\right)\Bigg|_{[0,1]}.\end{aligned}$$

Then for $j > 1$, we define collections of functions

$$\begin{aligned}\Phi_j : = \Big\{ & \sqrt{2^{j-1}}\alpha_4(2^j x+1)|_{[0,1]}, \sqrt{2^{j-1}}\alpha_1(2^j x-1), \sqrt{2^{j-1}}\alpha_2(2^j x-1), \\ & \sqrt{2^{j-1}}\alpha_3(2^j x-1), \sqrt{2^{j-1}}\alpha_4(2^j x-1), \sqrt{2^{j-1}}\alpha_1(2^j x-3), \\ & \sqrt{2^{j-1}}\alpha_2(2^j x-3), \ldots, \sqrt{2^{j-1}}\alpha_1(2^j x-2^j+1), \\ & \sqrt{2^{j-1}}\alpha_2(2^j x-2^j+1), \sqrt{2^{j-1}}\alpha_4(2^j x-2^j+1)|_{[0,1]}\Big\},\end{aligned}$$

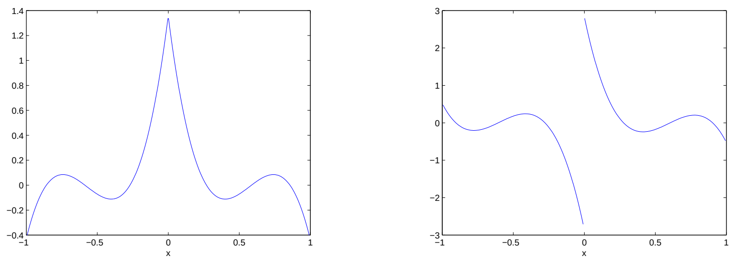



**Fig. 11** The second two dual basis functions $\beta_3$ and $\beta_4$.

and

$$\begin{aligned}\widetilde{\Phi}_j := \Big\{ & \sqrt{2^{j-1}}\beta_4(2^{j-1}x)\Big|_{[0,1]}, \sqrt{2^{j-1}}\beta_1(2^{j-1}x), \sqrt{2^{j-1}}\beta_2(2^{j-1}x),\\ & \sqrt{2^{j-1}}\beta_3(2^{j-1}x-1), \sqrt{2^{j-1}}\beta_4(2^{j-1}x-1), \sqrt{2^{j-1}}\beta_1(2^{j-1}x-1),\\ & \sqrt{2^{j-1}}\beta_2(2^{j-1}x-1), \ldots, \sqrt{2^{j-1}}\beta_1(2^{j-1}x-2^{j-1}+1),\\ & \sqrt{2^{j-1}}\beta_2(2^{j-1}x-2^{j-1}+1), \sqrt{2^{j-1}}\beta_4(2^{j-1}x-2^{j-1})\Big|_{[0,1]}\Big\}.\end{aligned}$$

From the construction directly follows that span $\Phi_j \subset V_j$, and span $\widetilde{\Phi}_j \subset \widetilde{V}_j$, respectively and we can check that $\#\Phi_j = \dim V_j$, $\#\widetilde{\Phi}_j = \dim \widetilde{V}_j$. Furthermore from the local supports and the normalization of the basis functions, one can easily verify that spectral radii of matrices $\langle \Phi_j, \Phi_j \rangle_{L_2(0,1)}$ and $\left\langle \widetilde{\Phi}_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)}$ are bounded uniformly in $j$ and then we have for any vector $\mathbf{c_j}$ of the appropriate size $||\mathbf{c_j}^T \Phi_j||_{L_2(0,1)} \lesssim ||\mathbf{c_j}||_{l_2}$ and $||\mathbf{c_j}^T \widetilde{\Phi}_j||_{L_2(0,1)} \lesssim ||\mathbf{c_j}||_{l_2}$.

$$\left\langle \Phi_2, \widetilde{\Phi}_2 \right\rangle_{L_2(0,1)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ \frac{-1}{14} & 0 & 0 & 0 & 1 & 0 & 0 & \frac{-1}{14} \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

**Fig. 12**

From the regular structure of matrices $\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)}$ (see Figure 12 and 13) immediately follows that their eigenvalues are contained in the interval $\left[\frac{1}{2} - \frac{1}{14}, 1 + \frac{2}{14}\right] = \left[\frac{3}{7}, \frac{8}{7}\right]$ and then inverse matrices exist with a uniformly bounded spectral norm.

$$\left\langle \Phi_3, \widetilde{\Phi}_3 \right\rangle_{L_2(0,1)} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ \frac{-1}{14} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 1 & 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & 0 & 1 & 0 & 0 & \frac{-1}{14} \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & 0 & \frac{-1}{14} & 0 & 0 & \frac{1}{2} \end{pmatrix}$$

**Fig. 13**

From this property and from $||\mathbf{c}_j^T \Phi_j||_{L_2(0,1)} \lesssim ||\mathbf{c}_j||_{l_2}$, we have from [13] that

$$\begin{aligned} ||\widetilde{\mathbf{c}}_j||_{l_2} \lesssim \left|\left| \left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)} \widetilde{\mathbf{c}}_j \right|\right|_{l_2} &= \sup_{\mathbf{c_j} \neq \mathbf{0}} \frac{\left| \left\langle \mathbf{c}_j, \left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)} \widetilde{\mathbf{c}}_j \right\rangle_{l_2} \right|}{||\mathbf{c}_j||_{l_2}} \\ &= \sup_{\mathbf{c_j} \neq \mathbf{0}} \frac{\left| \left\langle \mathbf{c}_j^T \Phi_j, \widetilde{\mathbf{c}}_j^T \widetilde{\Phi}_j \right\rangle_{L_2(0,1)} \right|}{||\mathbf{c}_j||_{l_2}} \lesssim ||\widetilde{\mathbf{c}}_j^T \widetilde{\Phi}_j||_{L_2(0,1)} \sup_{\mathbf{c_j} \neq \mathbf{0}} \frac{\left| |\mathbf{c}_j^T \Phi_j \right| |_{L_2(0,1)}}{||\mathbf{c}_j||_{l_2}} \\ &= \lesssim ||\widetilde{\mathbf{c}}_j^T \widetilde{\Phi}_j||_{L_2(0,1)}, \end{aligned}$$

and similarly from $||\widetilde{\mathbf{c}}_j^T \widetilde{\Phi}_j||_{L_2(0,1)} \lesssim ||\widetilde{\mathbf{c}}_j||_{l_2}$, we have $||\mathbf{c}_j||_{l_2} \lesssim ||\mathbf{c}_j^T \Phi_j||_{L_2(0,1)}$. Thus, we constructed uniform $L_2(0,1)-$Riesz bases $\Phi_j$ and $\widetilde{\Phi}_j$ for spaces $V_j$, and $\widetilde{V}_j$, respectively, such that $\left\langle \Phi_j, \widetilde{\Phi}_j \right\rangle_{L_2(0,1)}^{-1}$ exists with a uniformly bounded spectral norm. □

It remains to prove that constructed wavelets form uniform $L_2(0,1)-$Riesz bases for $W_j := V_j \cap \widetilde{V}_{j-1}^{\perp L_2(0,1)}$.

**Theorem 3** *Collections of functions $\widehat{\Psi}_1$ and $\Psi_j$ for $j > 1$ form uniform $L_2(0,1)-$Riesz bases for the space $W_1$ and $W_j$, respectively.*

**Proof.** Due to the orthogonality of functions from $\widehat{\Psi}_1$, they apparently form uniform $L_2(0,1)-$Riesz bases for $W_1$. For $j > 1$, we can numerically check that matrix

$$\langle \Psi_j, \Psi_j \rangle_{L_2(2^{-j-1},2^{-j})} := \begin{pmatrix} \frac{107179}{137200} & \frac{264659}{617400} & \frac{-1756}{5145} & \frac{19}{140} & \frac{-37501}{137200} & \frac{84241}{617400} \\ \frac{264659}{617400} & \frac{38971}{154350} & \frac{-620}{3087} & \frac{53}{630} & \frac{-84241}{617400} & \frac{578}{8575} \\ \frac{-1756}{5145} & \frac{-620}{3087} & \frac{704}{343} & 0 & \frac{1756}{5145} & \frac{-620}{3087} \\ \frac{19}{140} & \frac{53}{630} & 0 & \frac{13}{28} & \frac{19}{140} & \frac{-53}{630} \\ \frac{-37501}{137200} & \frac{-84241}{617400} & \frac{1756}{5145} & \frac{19}{140} & \frac{107179}{137200} & \frac{-264659}{617400} \\ \frac{84241}{617400} & \frac{578}{8575} & \frac{-620}{3087} & \frac{-53}{630} & \frac{-264659}{617400} & \frac{38971}{154350} \end{pmatrix},$$

which includes only wavelets with nonzero support in the interval $\left(2^{-j-1}, 2^{-j}\right)$, is positive definite. The same matrix will be obtained in any interval in the form $\left(k2^{-j-1}, (k+1)2^{-j-1}\right)$ for $k = 1, \ldots, 2^{j+1} - 2$. For $k = 1$ and $k = 2^{j+1} - 1$, we obtain similar matrix, where the first row and column will be deleted for $k = 1$ and the fifth row and column for $k = 2^{j+1} - 1$. These smaller matrices are also positive definite. Consequently any matrix $\langle \Psi_j, \Psi_j \rangle_{L_2(0,1)}$ can be composed from these small matrices and its the smallest eigenvalue can be bounded by the smallest eigenvalue of the small matrix and the largest eigenvalue can be bounded by double of the largest eigenvalue of the small matrix. Then by using the same arguments as in the last paragraph of the previous proof, we can conclude that collections of functions $\Psi_j$ form uniform $L_2(0,1)-$Riesz bases for the spaces $W_j$ for $j > 1$. □

Now, we can apply Theorem 1. It is well-known [7] that a direct estimate of order $d$ is satisfied when all polynomials of order $d$ satisfying possibly boundary conditions are included in the space $V_{j_0}$, while an inverse estimate of order $\gamma$ is known to hold with $\gamma = r + \frac{3}{2}$ when spaces $V_j$ are spanned by piecewise smooth $C^r(0,1)$ functions for some $r \in \{-1, 0, 1, \ldots\}$, where $r = -1$ means that no global continuity is satisfied. For constructed basis, we have $d = \widetilde{d} = 4$, $\gamma = \frac{5}{2}$, and $\widetilde{\gamma} = \frac{1}{2}$. Then Theorems 1, 2, 3 imply the following results.

**Theorem 4** *Let* $\mathcal{H}^s(0,1) = \left[L_2(0,1), H^d(0,1) \cap H_0^1(0,1)\right]_{s/4}$, *for* $s \in [0,4]$ *and* $\mathcal{H}^s(0,1) := \left(\mathcal{H}^{-s}(0,1)\right)'$ *for* $s < 0$. *Then for* $s \in (-\frac{1}{2}, \frac{5}{2})$, *the collection* $\widehat{\Phi}_1 \cup 2^{-s}\widehat{\Psi}_1 \bigcup_{j=2}^{\infty} 2^{-sj}\Psi_j$ *is a Riesz basis for* $\mathcal{H}^s(0,1)$.

Especially, the constructed basis, when normalized in $L_2(0,1)$ or $H^1(0,1)$, forms a Riesz basis for $L_2(0,1)$ and $H^1(0,1)$, respectively.

## 4 Condition numbers

In this section, we provide condition numbers of one-dimensional stiffness matrices $\mathbf{S}$ and condition numbers of mass matrices $\mathbf{M}$ (see (7)) for different decomposition levels. Basis functions are normalized in $L_2(0,1)$ or in $H^1(0,1)$, respectively:

$$\mathbf{S} = \left[ \frac{\int_0^1 \frac{\partial \psi_\lambda}{\partial x} \frac{\partial \psi_\mu}{\partial x} \, dx}{|\psi_\lambda|_{H^1(0,1)} \, |\psi_\mu|_{H^1(0,1)}} \right]_{\lambda,\mu \in \mathcal{J}} \qquad \text{and} \qquad \mathbf{M} = \left[ \frac{\int_0^1 \psi_\lambda \, \psi_\mu \, dx}{\|\psi_\lambda\|_{L_2(0,1)} \|\psi_\mu\|_{L_2(0,1)}} \right]_{\lambda,\mu \in \mathcal{J}}$$

And we compare condition numbers with condition numbers for a similar wavelet basis proposed in [13]. Results are summarized in Table 1.

| | DS | | NEW | |
|---|---|---|---|---|
| n | COND $L_2$ | COND $H^1$ | COND $L_2$ | COND $H^1$ |
| 4 | 7.0 | 1.7 | 1.0 | 2.6 |
| 8 | 15.2 | 4.4 | 1.0 | 2.9 |
| 16 | 24.3 | 5.5 | 6.3 | 3.7 |
| 32 | 32.0 | 5.8 | 12.7 | 4.4 |
| 64 | 37.3 | 6.2 | 18.8 | 4.8 |
| 128 | 41.2 | 6.6 | 24.4 | 5.1 |
| 256 | 44.1 | 6.8 | 29.3 | 5.3 |
| 512 | 46.3 | 6.9 | 33.6 | 5.4 |
| 1024 | 48.1 | 7.0 | 37.2 | 5.5 |
| 2048 | 49.5 | 7.1 | 40.4 | 5.5 |
| 4096 | 50.7 | 7.1 | 43.1 | 5.5 |

**Table 1** Condition numbers of matrices **M** and **S**.

Further, we consider here the following Dirichlet problem

$$-\sum_{i=1}^{d} \frac{\partial^2 u}{\partial x_i^2} = f \quad \text{in} \quad \Omega = (0,1)^d \qquad \text{with} \quad u = 0 \quad \text{on} \quad \partial\Omega$$

for given $f \in H^{-1}(\Omega)$ in two and three dimensions. A Riesz wavelet basis for $H_0^1(\Omega)$ can be constructed by a tensor product of univariate Riesz wavelet bases. We consider here two options: an isotropic and an anisotropic tensor product. Isotropic wavelets arise as a tensor product of univariate wavelets and scaling functions from the same decomposition level. Then e.g. in two dimensions, we have these three types of wavelets

$$\phi_{j,k} \otimes \psi_{j,l}, \quad \psi_{j,k} \otimes \phi_{j,l}, \quad \psi_{j,k} \otimes \psi_{j,l},$$

where $\phi_{j,k}$ is a scaling function on the level $j$ and $\psi_{j,l}$ is a wavelet on the same level. For a definition in arbitrary dimensions, we refer to [17]. Anisotropic wavelets were already introduced in (4). Then e.g. in two dimensions, wavelets will be of the form

$$\psi_{j,k} \otimes \psi_{l,m},$$

where $\psi_{j,k}$ and $\psi_{l,m}$ are wavelets generally on different levels. Therefore their supports can be arbitrarily anisotropic. In all cases, we use a normalization of basis functions in $H^1-$ seminorm. In Tables 2 and 3, we summarize condition numbers of stiffness matrices in two and three dimensions. We again compare them with condition numbers for a similar wavelet basis proposed in [13].

| | DS | | NEW | |
|---|---|---|---|---|
| n | isotropic | anisotropic | isotropic | anisotropic |
| 16 | 11.7 | 11.7 | 2.6 | 2.6 |
| 64 | 57.8 | 57.8 | 2.9 | 2.9 |
| 256 | 82.8 | 103.9 | 28.4 | 16.7 |
| 1024 | 90.2 | 144.0 | 49.8 | 43.4 |
| 4096 | 94.0 | 180.4 | 59.8 | 68.5 |
| 16384 | 95.4 | 213.4 | 66.1 | 92.9 |
| 65536 | 95.9 | 239.2 | 69.8 | 117.1 |
| 262144 | 96.1 | 259.6 | 72.2 | 138.7 |
| 1048576 | 96.2 | 281.0 | 73.7 | 158.1 |

**Table 2** Condition numbers of stiffness matrices for $d = 2$.

In all tables, $n$ represents the number of basis functions, NEW denotes new wavelets, and finally DS denotes wavelets proposed in [13]. Obtained results confirm that condition numbers of stiffness matrices are on the first two decomposition levels small and independent of the spatial dimension.

| | DS | | NEW | |
|---|---|---|---|---|
| n | isotropic | anisotropic | isotropic | anisotropic |
| 64 | 81.7 | 81.7 | 2.6 | 2.6 |
| 512 | 812.0 | 812.0 | 2.9 | 2.9 |
| 4096 | 1383.0 | 2329.0 | 366.7 | 100.3 |
| 32768 | 1537.5 | 4297.5 | 764.3 | 517.8 |
| 262144 | 1595.8 | 6147.1 | 1022.0 | 1212.6 |
| 2097152 | 1611.3 | 7994.3 | 1159.2 | 2125.2 |

**Table 3** Condition numbers of stiffness matrices for $d = 3$.

## References


1. Bramble, J. H., Cohen, A., Dahmen, W.: Multiscale Problems and Methods in Numerical Simulations, Springer-Verlag Berlin Heidelberg (2003).
2. Černá, D., Finěk, V.: Construction of optimally conditioned cubic spline wavelets on the interval. Adv. Comput. Math. 34, 219–252 (2011).
3. Černá, D., Finěk, V.: Cubic Spline Wavelets with Complementary Boundary Conditions. Appl. Math. Comput. 219, 1853–1865 (2012).
4. Černá, D., Finěk, V.: Approximate Multiplication in Adaptive Wavelet Methods, Cent. Eur. J. Math. 11, 972–983 (2013).
5. Černá, D., Finěk, V.: Quadratic Spline Wavelets with Short Support for Fourth-Order Problems. Result. Math. 66, 525–540 (2014).
6. Černá, D., Finěk, V.: Cubic Spline Wavelets with Short Support for Fourth-Order Problems. Appl. Math. Comput. 243, 44–56 (2014).
7. Ciarlet, P.: Finite Element Method for Elliptic Problems, Society for Industrial and Applied Mathematics. Philadelphia, PA, USA (2002).
8. Cohen, A., Dahmen, W., DeVore, R.: Adaptive Wavelet Schemes for Elliptic Operator Equations - Convergence Rates. Math. Comput. 70, 27–75 (2001).
9. Cohen, A., Dahmen, W., DeVore, R.: Adaptive Wavelet Methods II - Beyond the elliptic case. Found. Math. 2, 203–245 (2002).

10. Dahmen, W.: Stability of multiscale transformations. J. Fourier Anal. Appl. 2, 341–361 (1996).
11. Dahmen, W., Stevenson R.: Element-by-element construction of wavelets satisfying stability and moment conditions. SIAM J. Numer. Anal. 37, 319–352 (1999).
12. Dijkema, T. J., Schwab, Ch., Stevenson, R.: An adaptive wavelet method for solving high-dimensional elliptic PDEs. Constr. Approx. 30, 423–455 (2009).
13. Dijkema, T. J., Stevenson, R.: A sparse Laplacian in tensor product wavelet coordinates. Numer. Math. 115, 433–449 (2010).
14. Griebel, M., Oswald, P.: Tensor product type subspace splittings and multilevel iterative methods for anisotropic problems. Adv. Comput. Math. 4, 171–206 (1995).
15. Primbs, M.: New stable biorthogonal spline-wavelets on the interval. Result. Math. 57, 121–162 (2010).
16. Stevenson, R.: Adaptive solution of operator equations using wavelet frames. SIAM J. Numer. Anal. 41, 1074–1100 (2003).
17. Urban, K.: Wavelet Methods for Elliptic Partial Differential Equations. Oxford University Press, Oxford (2009).