M U N I

HABILITATION THESIS REVIEWER'S REPORT

Masaryk University	
Faculty	**Faculty of Science**
Procedure field	**Mathematics - Geometry**
Applicant	**Yaroslav Bazaykin, Ph.D., D.Sc.**
Applicant's home unit, institution	**Masaryk University** **Faculty of Science**
Habilitation thesis	**Noncompact Riemannian manifolds with special holonomy**
Reviewer	**Prof. Dr. Lorenz Schwachhoefer**
Reviewer's home unit, institution	**TU Dortmund University** **Faculty for Mathematics**

See attached document.

Reviewer's questions for the habilitation thesis defence (number of questions up to the reviewer)

Is there any possibility to use your methods to construct examples of special Kähler metrics on orbifolds of arbitrary (even) dimension?

Conclusion

The habilitation thesis entitled “**Noncompact Riemennian manifolds with special holonomy**” by **Yaroslav Bazaykin** **fulfils** **–** **~~does not fulfil~~** the requirements expected of a habilitation thesis in the field of Mathematics – Geometry.

Date: Dortmund, September 28, 2020 Signature:

technische universität dortmund

fakultät für mathematik

Lehrstuhl für Differentialgeometrie
Prof. Dr. L. Schwachhöfer

TU Dortmund - Vogelpothsweg 87 - 44221 Dortmund

Durchwahl (0231)7 55-3094/5
Telefax (0231)7 55-5937
e-mail:
lorenz.schwachhoefer@math.uni-dortmund.de

Dortmund, 28. September 2020

Letter of expertise on the Habilitation thesis of Yaroslav Bazaykin

Noncompact Riemannian manifolds with special holonomy

The investigation of manifolds with one of the exceptional holonomies G_2 and Spin(7) is a rapidly developing area in modern geometry and topology and in mathematical physics, particularly in relation to string theory, M-theory and mirror symmetry. After the (local) existence of manifolds with these holonomies was settled by Bryant in 1987, complete examples were found by Bryant and Salamon (1989). A further big break-through was the construction of closed G_2- and Spin(7)-manifolds by Joyce (1996); later another consructution method of compact G_2-manifolds was established by Kovalev (2003), but since then not much progress was made in constructing new closed examples. Indeed, the constructions of Joyce and Kovalev still restult only in a finite number of 7- or 8-dimensional manifolds with holonomy G_2 and Spin(7), respectively. So far, it is even unclear if the number of homotopy types of (closed) manifolds admitting a torsion free G_2- or Spin(7)-structure is finite or infinite.

The research of Yaroslav Bazaykin has been mainly concerned with the construction of *non-compact* examples of G_2- or Spin(7)-manifolds, as well as manifolds with other special holonomy groups. In comparison to the construction of *compact* examples, this is easier at first glance in the sense that in many cases, the differential equations which occur here reduce to ODEs, so that the solvability is easily established. On the other hand, it is more difficult, as for the construction of *compact* G_2- or Spin(7)-manifolds, very powerful results are available such as the solvability of the Calabi-Yau-problem or the deformation method of Joyce which allows to deform a G_2- or Spin(7)-structure with "sufficiently little" torsion into a torsion free one. None of these general results carries over to the non-compact case, so that different methods have to be developed in order to find new examples.

The qualitative understanding of non-compact solutions of manifolds with special holonomy is very useful for several reasons. The first is that the glueing constructions on *closed* manifolds with special holonomy relies on the glueing of non-compact examples along their "asymptotically flat" boundary; this happens e.g. by replacing a neighborhood of a singularity of a torus quotient by a finite group by the conormal bundle of (e.g.) a sphere with the Eguchi-Hensen metric (the method used by Joyce to find closed examples) or glueing asymptotically flat bundles over $K3$-surfaces (the method used by Kovalev). In either case, it is crucial that the qualitative behavior of the metrics of the (non-compact) parts used in this construction is well understood, and having more examples will possibly result in further glueing methods, resulting in the construction of further closed examples. Another motivation of finding non-compact manifolds with special holonomy is that in Mathematical Physics the models used in string theory, M-theory and mirror symmetry motivate explicit descriptions of non-compact manifolds (and orbifolds) with special holonomy and prescribed asymptotic behaviour.

In the sequel I shall give a more detailed account on the papers which are included in this habilitation thesis. The references [1] and [2] deal with the construction of complete examples of manifolds with holonomy Spin(7) using 3-Sasakian geometry. The idea can be described as follows. One starts with a 3-Sasakian 7-manifold (M^7, g), whose 3-Sasaki-structure is induced by a locally free G-action, where G is either $Sp(1)$ or $SO(3) = Sp(1)/\mathbb{Z}_2$. The orbit space $\mathcal{O}^4 = M^7/G$ is then an orbifold, and $M^7 \to \mathcal{O}^4$ is a principal G-bundle over $\mathcal{O}^4$. Moreover, this projection submerses to a quaternionic-Kähler metric on $\mathcal{O}^4$.

On the other hand, the metric cone $\bar{M} := (M \times (0,\infty), dr^2 + r^2 g)$ over a 3-Sasakian manifold is hyper-Kähler, i.e., has holonomy $SU(4)$, and there is an induced $\mathbb{C}^*$-action whose quotient $\mathcal{Z}^6$ is a 6-dimensional orbifold, so that $\bar{M} \to \mathcal{Z}^6$ is a principal $\mathbb{C}^*$-bundle; again, the metric on $\bar{M}$ submerses to a Kähler-Einstein metric on $\mathcal{Z}^6$. Therefore, one can consider the associated vector bundles

$$\mathcal{M}_1^8 := M^7 \times_G \mathbb{R}^4 \longrightarrow \mathcal{O}, \qquad \mathcal{M}_2^8 := \bar{M} \times_{\mathbb{C}^*} \mathbb{C} \longrightarrow \mathcal{Z}^6.$$

Then one can describe Riemannian metrics on $\mathcal{M}_1^8$ and $\mathcal{M}_2^8$ by four "weight functions" $A_1, A_2, A_3, B : [0,\infty) \to \mathbb{R}$ and show that this metric has holonomy contained in Spin(7) iff these functions satisfy a certain system of ordinary differential equations. In fact, choosing the initial conditions at $t = 0$ carefully, these metrics induce orbifold metrics on $\mathcal{M}_1^8/\mathbb{Z}_p$ and $\mathcal{M}_2^8/\mathbb{Z}_p$ for appropriate values of p, which are *complete* orbifold metrics with holonomy Spin(7), and there is an asymptotically well controlled splitting into a product metric.

As an explicit example, in [3] he considers the Aloff-Wallach space $M^7 := SU(3)/U(1)_{1,1,-2}$ which is an explicit example of a 3-Sasakian manifold. By carefully examining the orbifold metrics associated to this by the techniques in [1] and [2], the authors obtain a 1-parameter family of complete Riemannian metrics (g_α), $0 \leq \alpha \leq 1$, on the cone $M^8 := M^7 \times (0,\infty)$ which are Calabi-Yau manifolds, i.e., have holonomy $SU(4) \subset \mathrm{Spin}(7)$ for $0 \leq \alpha < 1$, and hyper-Kähler, i.e., with holonomy $Sp(2) \subset SU(4) \subset \mathrm{Spin}(7)$ at $\alpha = 1$. This family connects the Calabi metrics on the line bundle of degree p over $\mathcal{Z}^6$ at $\alpha = 0$ to the hyper-Kähler metric on $T^*\mathbb{CP}^2$ at $\alpha = 1$, also constructed by Calabi. Furthermore, the examples constructed in [1] and [2] are extended to the orbifold case, and it is investigated when they have holonomy Spin(7) and in which case the holonomy further reduces to $SU(4) \subset \mathrm{Spin}(7)$, i.e., when this metric is a Calabi-Yau metric.

In the reference [4], he turns to the construction of metrics with holonomy G_2. As before, the starting point is a 3-Sasakian manifold, and here, one assumes that the quotient $\mathcal{O}^4 = M^7/G$ is a Kähler manifold. Then one considers metrics on the associated vector bundles

$$\mathcal{N}^7 := M^7 \times_G \mathbb{R}^3 \longrightarrow \mathcal{O}^4.$$

Again, he now describes bundle metrics on $\mathcal{N}^7$ which depend on three functions in one variable, and he shows that these metrics have holonomy G_2 iff these functions satisfy a certain system of ordinary differential equations. As an important example, he gives complete G_2-metric on such bundles where $\mathcal{O}^4$ is a weighted projective space $\mathbb{CP}^2(q_1, q_2, q_3)$ with certain coefficients q_i and hence obtains an explicit construction of G_2-metrics on such bundles.

The references [5] and [6] are concerned with the construction of explicit complete Ricci flat metrics on 4-dimensional orbifolds. Most of the complete non-compact examples of Ricci flat manifolds have an isometry group which acts on them with cohomogeneity one; mostly these are homogeneous vector bundles. For instance, the Eguchi-Hansen metric on T^*S^2 is an example of such a metric.

In these references, new complete examples of cohomogeneity two Ricci flat orbifolds are constructed; in fact, the examples in [5] are labelled as (M_a, g_a) for $a = (-1, 1) \cap \mathbb{Q}$; only for $a = 0$ are these metrics of cohomogeneity one. In fact, each M_a is a fibration with $\mathbb{C}$-fiber over the orbifold $S^2(k, l)$ for appropriate integers k, l depending on a, where $S^2(k, l)$ denotes the 2-sphere with two conical orbifold singularities with respective cone angles $2\pi/k$ and $2\pi/l$. These examples include the complex cotangent bundle $T^*S^2(k, l)$, thus generalizing the Eguchi-Hansen metric to the orbifold case. However, as the author shows in [6], these examples do not have full holonomy, but they are special Kähler surfaces, i.e., they have holonomy $SU(2)$.

While in [5] the metrics are determined by explicit coordinate expressions, the discussion in [6] presents a more conceptual picture. Namely, one realizes these as a generalization of the Page construction which starts by considering the quotient $T^4/\mathbb{Z}_2$ and then resolving the 16 orbifold singularities of this space which yields a $K3$-surface. The generalization now consists in considering quotients $T^4/\mathbb{Z}_p$ and then resolving the singularities adapting Page's techniques. Remarkably, this only works in case $p = 3$, and from this, he is able to construct an 58-dimensional family of metrics on a $K3$-surface with holonomy $SU(2)$.

Summary. Having elaborated on the various contributions, I consider this Habilitation thesis to be an impressive work of mathematics. The presentation of the introduction is a very well readable introduction into the field, giving an excellent summary of the background material needed to appreciate the reading of the articles [1] – [6]. In my opinion, this thesis demonstrates technical strength and the ability to carry out good independent research in an important area of modern geometry. Therefore, I consider that Yaroslav Bazaykin should be granted the Habilitation.

(Prof. Dr. Lorenz Schwachhöfer)