Habilitation Thesis Reviewer's Report

Masaryk University

Faculty: Faculty of Science

Procedure field: Mathematics - Mathematical Analysis

Applicant: RNDr. Martina Pavlačková, Ph.D.

Habilitation thesis: Boundary value problems for second-order differential inclusions on compact and non-compact intervals in the Euclidean and abstract spaces

Reviewer: Irene Benedetti, Ph.D

Reviewer's home unit, institution: Department of Mathematics and Computer Science, University of Perugia

The habilitation thesis of Martina Pavlačková is devoted to the study of boundary value problems for second order partial differential inclusions. Specifically, in the twelve research papers presented she considers vector problems on compact and non-compact intervals, vector problems with impulses and problems in Banach spaces. She begins considering the following problem in $\mathbb{R}^n$

$$\begin{cases} \ddot{x} \in F(t, x(t), \dot{x}(t)), & \text{for a.a.} \quad t \in I \\ x \in S \end{cases}$$

where

- $I \subset \mathbb{R}$ is a possibily non-compact interval,
- $S \subset AC^1_{loc}$, where AC^1_{loc} denotes the space of functions $x : I \to \mathbb{R}^n$ with locally absolutely continuous first derivative, and
- $F : I \times \mathbb{R}^n \times \mathbb{R}^n \multimap \mathbb{R}^n$ is an upper-Carathéodory mapping.

The above differential problem is then studied with impulses and subsequently in Banach spaces, i.e., with the multimap $F : I \times E \times E \multimap E$ being an upper-Carathéodory mapping, with E a separable Banach space satisfying the Radon-Nykodim property.

The research of Martina Pavlačková has focused on the study of Floquet problems, applicable to the analysis of many interesting phenomena arising from life sciences. The articles presented develop several existence results, applying suitable continuation principles. The continuation principle consists into associating to the problem studied a one-parameter family of linearized problems, the parameter is a real number $\lambda \in [0, 1]$ such that at the value $\lambda = 1$ one recovers the original problem; into proving that each problem has a solution and into constructing an homotopy linking the solutions of the problem corresponding to the parameter equal to zero and equal to one. The starting point to do so is Theorem 2.1 in the thesis, developed by the canditate in collaboration with Professor Jan Andres and presented in reference [23] of the thesis. This result is subsequently extended to the construction of continuation principles applicable to infinite dimensional and impulse problems. The results are also extended to the challenging problem of analyzing the topological structure of the solution set.

A main difficulty in this kind of results lies in proving the transversality condition:

- the solution map $T(\cdot, \lambda)$, $\lambda \in [0,1]$, has no fixed points on the boundary ∂Q of a prescribed set $Q \subset C^1(I, \mathbb{R}^n)$.

In her papers, Martina Pavlačková has investigated this condition in depth, proving sufficient conditions that imply the transversality condition while being much easier to verify. She developed a bound set approach for second order problems. A set $K \subset E$ is said to be a bound set if the trajectory of any solution of the problem which is entirely contained in K remains for any time inside K. Note that this is a weaker requirement than the invariance of the set K. Martina Pavlačková prove the existence of a bound set for second order differential problems, by means of a bounding function, i.e. a function $V : E \to \mathbb{R}$ such that $V(x) = 0$ for any point $x \in \partial K$ and $V(x) \leq 0$ for any point $x \in \overline{K}$. In particular, she proves that K is a bound set assuming a transversality condition of the following type:

$$\langle \nabla V(x), w \rangle > 0 \text{ and } \langle \ddot{V}_x(v), v \rangle \geq 0 \tag{1}$$

for a.a. $t \in I$, $\forall\, x \in \partial K,\ v \in E,\ w \in F(t, x, v)$.

The transversality condition (1) has been carefully studied considering various cases of regularity of the multimap F present in the differential problems and of the bounding function itself. The result contained in papers [14], [16], [17], [18], [73], [74] are particularly interesting, since the transversality condition (1) is localized on the boundary of K. This is extremely significant since the transversality condition (1) required for any x in a whole neighborhood of the boundary of K implies not only that K is a bound set, but also that K is a positively invariant set, i.e. a much stronger assumption.

The thesis is well written and well organized. It includes an extensive historical introduction, in which the original results obtained by Martina Pavlačková are compared with the existing literature on the subject. Subsequently, for each problem investigated there is a brief but effective introduction, which includes an exposition of simple real-life models that lead to the abstract problem studied. Next, Martina Pavlačková describes the continuation principle used, highlighting the novelties with respect to the literature and the difficulties of application.

The articles in the thesis, and the presentation that relates them, show a line of research that is profound, rigorous, and of high value, showing beyond doubt that Martina Pavlačková is a talented scholar in her full maturity.

Conclusion

The habilitation thesis entitled "Boundary value problems for second-order differential inclusions on compact and non-compact intervals in the Euclidean and abstract spaces" by Martina Pavlačková fulfils requirements expected of a habilitation thesis in the field of Mathematics - Mathematical Analysis.

Date: 6 December 2020 **Signature:**