Referee's report on the habilitation thesis

Masaryk University	
Faculty	of Science
Habilitation field	Mathematics – mathematical analysis
Applicant	RNDr. Michal Veselý, Ph.D.
Affiliation	Masaryk University, Faculty of Science
Habilitation thesis	Solution spaces of almost periodic homogeneous linear difference and differential systems
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Text of the report

The Habilitation Thesis by Michal Veselý is devoted to studying almost periodic linear homogeneous difference and differential systems.

It is well known that bounded solutions of linear periodic systems are almost periodic. For linear systems with almost periodic coefficients this statement is not true in general.

The main goal of the thesis is to solve the following problems: if subsets of systems with non almost periodic solutions and systems with almost periodic solutions are dense in space of all linear almost periodic systems.

These problems are closely connected with questions on reducibility and non-reducibility of linear almost periodic systems to systems with constant matrices and ergodicity of corresponding linear skew-product flows.

Consider the achievements of the thesis in more details.

1. Several theorems about constructing almost periodic sequences in pseudo-metric space with some prescribed properties are proved. In the next sections, these results are used to prove theorems on density of systems with non almost periodic solutions. Also the author constructs an almost periodic homogeneous linear difference system with coefficients belonging to a pseudo-metric space which does not have any non-trivial almost periodic solution.

2. The author proposes two approaches to the study of systems of linear almost periodic difference equations

$$x_{n+1} = A_n x_n, n \in \mathbb{Z}, \qquad (1)$$

in a general statement: matrices A_n belong to some group X of $m \times m$ matrices $\mathcal{M}(m, F)$ over infinite field F.

First, the concepts of transformable and weakly transformable groups of matrices are introduced. These properties characterize a certain "uniformity" of matrix group. Author analyzes in detail several examples of transformable and weakly transformable groups. Such important groups as the group of unitary matrices and the group of orthogonal matrices with determinant 1 are transformable and the group of all real orthogonal matrices is weakly transformable. The concepts of transformable and weakly transformable groups of matrices are very fruitful to describe groups X satisfying following property: systems (1) with almost periodic sequences of matrices $A_n \in X$ and having no non-trivial almost periodic solutions form a dense subset in the set of all systems (1) with almost periodic sequences of matrices $A_n \in X$.

3. The second approach is associated with the so-called properties P and P^*. This made it possible to study the set of systems with limit-periodic coefficients. In any neighborhood of any system with limit-periodic coefficients there is a system with limit-periodic coefficients and without non-zero asymptotically almost periodic solutions (and hence, a system without non-zero almost periodic solutions).

Separately, the case of commutative and unbounded groups is considered. The author obtains conditions that in any neighborhood of any system there exists a system with not almost periodic solution of given initial value problem.

4. It is proved that, for any countable and totally bounded subset of pseudo-metric space, there exists a limit periodic sequence whose range is this set and each value from the set is repeated periodically in the sequence. If additionally subset is dense in itself, then there exists a limit periodic sequence such that each value from the set is included in the sequence only once.

These results are of independent interest. And they are used by proving theorems about density of systems without almost periodic solutions.

Analogous results are obtained for functions.

I note interesting corollary: let X be a bounded group. Then there exists an almost periodic sequence $\{C_n\}$ satisfying $cls\{C_k, k \in Z\} = X$ such that all solutions of the system $y_{n+1} = C_n y_n$ are almost periodic (see Theorem 3.11).

5. Author consider systems of m homogeneous linear differential equations

$$x'(t) = A(t)x(t), t \in \mathbb{R}, \tag{2}$$

where A is an almost periodic function of skew-Hermitian or skew-symmetric matrices.

It is known that the set of systems, whose all solutions are almost periodic, is dense in the set of all systems with almost periodic skew-Hermitian or skew-symmetric matrices. In the thesis, it is proved that the complementary subset of systems is also dense: in any neighborhood of any system with almost periodic skew-Hermitian matrix there exists a system, which does not have an almost periodic solution other than the trivial one. Separately, analogous result is proved for linear almost periodic systems with skew-symmetric matrices in Euclidian space.

For proving these statements the author uses his method of constructing of almost periodic functions with prescribed properties. Note that the method is applied for functions with values in general pseudo-metric space.

Referee's Question. Compare the results of the thesis with similar problems for following systems: in product of d-dimensional torus $\mathbb{T}^d = \mathbb{R}^d/\mathbb{Z}^d$ and some group of n-dimensional matrices G let consider a diffeomorphism $(\alpha, A) : (x, y) \to (x + \alpha, A(x)y)$. The iterations of (α, A) define a discrete-time quasi-periodic system. Let α be fixed. Is a subset of systems with non almost periodic solutions (a subset of systems with almost periodic solutions) dense in the set of all systems defined by mappings $A : \mathbb{T}^d \to G$?

Conclusion

The author has obtained new original results in the theory of almost periodic equations. The thesis is well and carefully written. All proofs are correct. The main results of the thesis are published in good international journals.

Habilitation thesis by Dr. M.Veselý "Solution spaces of almost periodic homogeneous linear difference and differential systems" **meets** the standard requirements for habilitation theses in mathematical analysis.

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